Numerical methods for relativistic HD and MHD

José Antonio Font
Departamento de Astronomía y Astrofísica
VNIVERSITAT DE VALÈNCIA

Outline

Lecture 1: Relativistic hydrodynamics and MHD

Lecture 2: Numerical methods (Riemann solvers)

Homework

Further reading (HD & MHD)

A.M. Anile, "Relativistic fluids and magneto-fluids", Cambridge University Press (1989)

L. Rezzolla & O. Zanotti, "Relativistic hydrodynamics", Oxford University Press (2013)

J.M. Martí & E. Müller, "Numerical hydrodynamics in special relativity", Living Reviews in Relativity (2003)

J.M. Martí & E. Müller, "Grid-based methods in relativistic hydrodynamics and magnetohydrodynamics", Living Reviews in Computational Astrophysics (2015)

J.A. Font, "Numerical hydrodynamics and magnetohydrodynamics in general relativity", Living Reviews in Relativity (2008)

F. Banyuls et al, "Numerical 3+1 general relativistic hydrodynamics: a local characteristic approach", Astrophysical Journal, 476, 221 (1997)

L. Antón et al, "Numerical 3+1 general relativistic magneto-hydrodynamics: a local characteristic approach", Astrophysical Journal, 637, 296 (2006)

Further reading (Methods, NR, ET)

- R. Leveque, "Nonlinear conservation laws and finite volume methods for astrophysical fluid flow", Springer (1998)
- E. Toro, "Riemann solvers and numerical methods for fluid dynamics", Springer (1997)

- M. Alcubierre, "Introduction to 3+1 Numerical Relativity", Oxford (2007)
- T. Baumgarte & S. Shapiro, "Numerical relativity: solving Einstein's equations on the computer", Cambridge University Press (2010)

- F. Löffler et al, "The Einstein Toolkit: A community computational infrastructure for relativistic astrophysics", CQG, 29, 115001 (2012)
- P. Mösta et al, "GRHydro: A new open source general-relativistic magnetohydrodynamics code for the Einstein toolkit", CQG, 31, 015005 (2014)

Lecture 1
Hydrodynamics and MHD

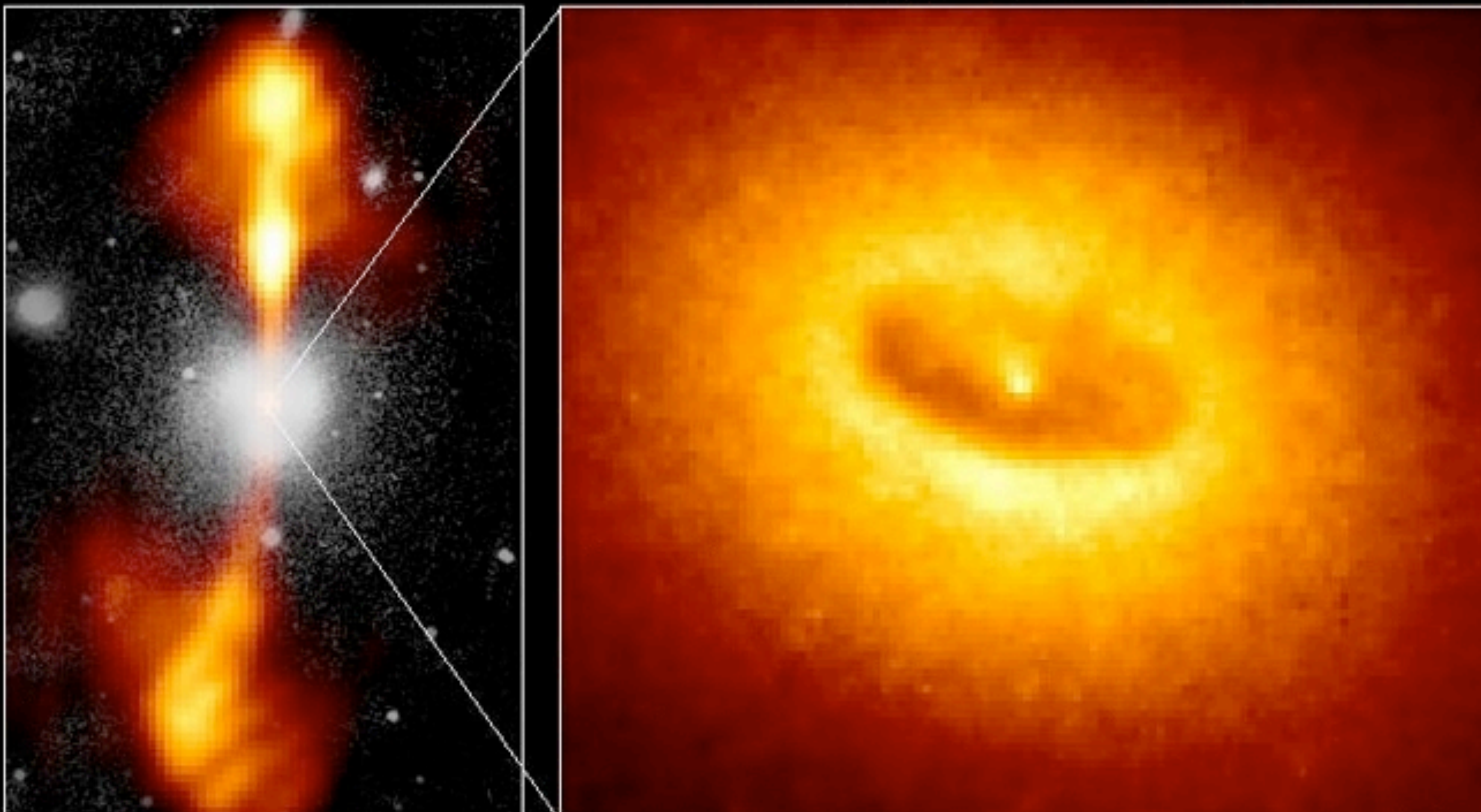
Core of Galaxy NGC 4261

Hubble Space Telescope

Wide Field / Planetary Camera

Ground-Based Optical/Radio Image

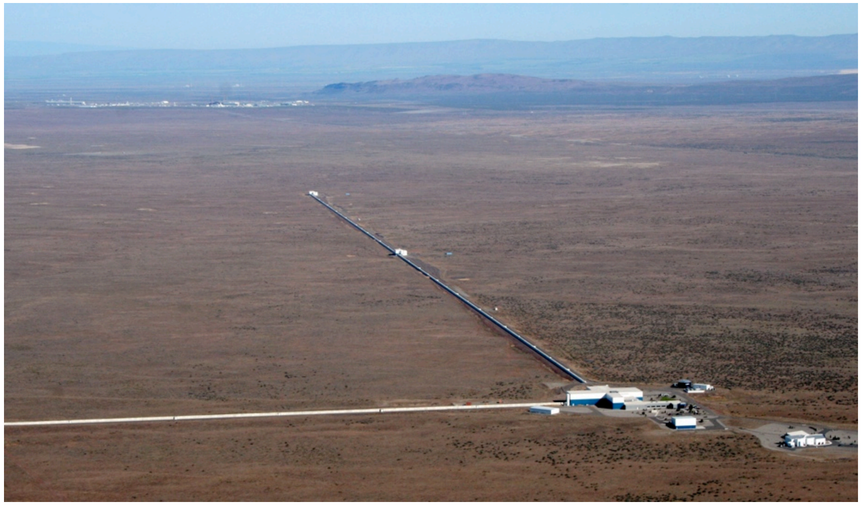
HST Image of a Gas and Dust Disk



380 Arc Seconds
88,000 LIGHT-YEARS

1.7 Arc Seconds
400 LIGHT-YEARS

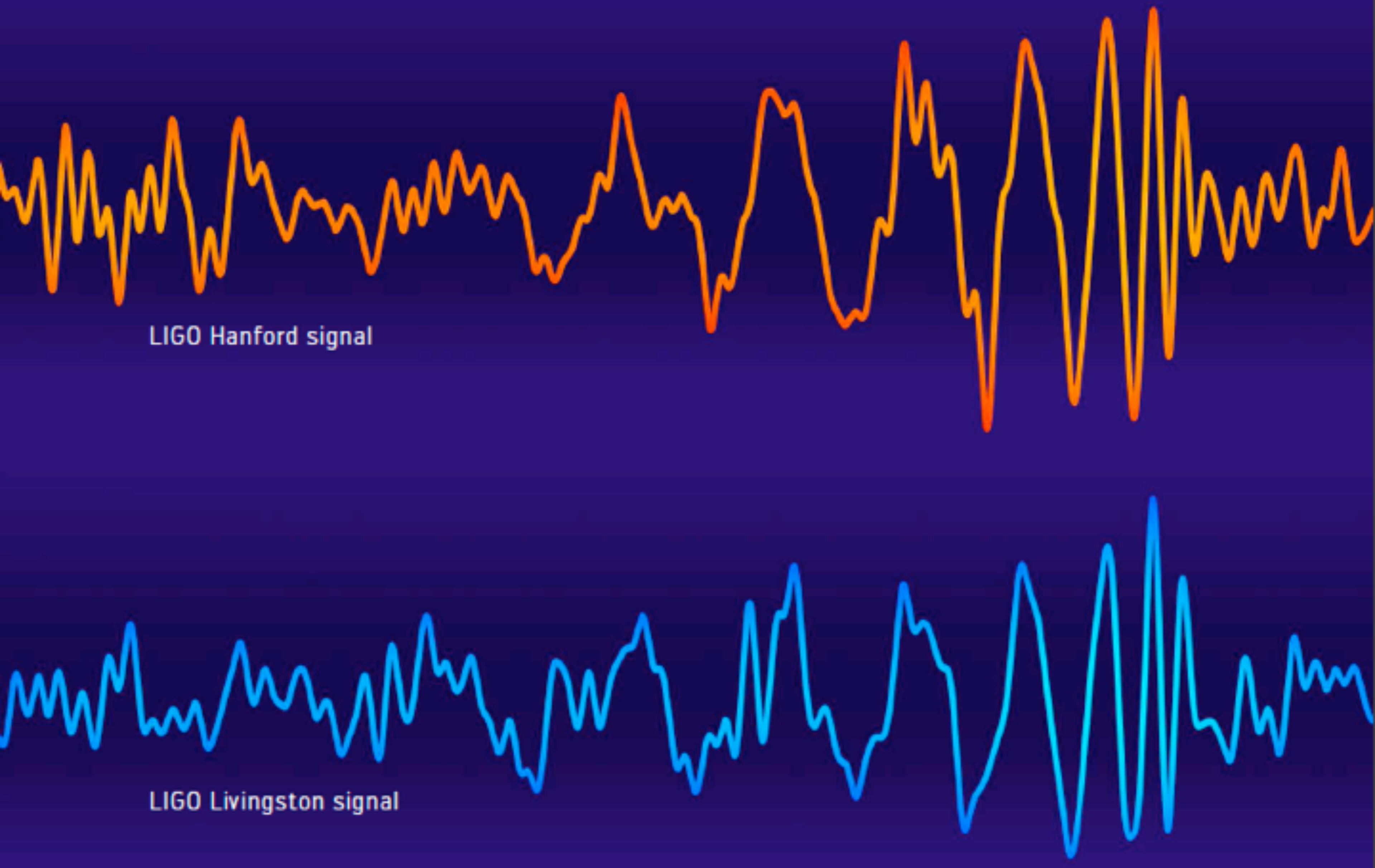
Numerical Relativity provides gravitational radiation waveforms: main historical motivation driving the development of this area of General Relativity.



GW150914

First detection!

9:50:45 UTC, 14 September 2015



General relativity and relativistic hydrodynamics play a major role in the description of gravitational collapse leading to the formation of compact objects (neutron stars and black holes).

Prime Sources of Gravitational Radiation.

Time-dependent evolutions of fluid flow coupled to the spacetime geometry (Einstein's equations) possible through accurate, large-scale numerical simulations.

Some scenarios can be described in the test-fluid approximation: GRHD/GRMHD computations in curved backgrounds.

GRHD/GRMHD equations are nonlinear hyperbolic systems. Solid mathematical foundations and accurate numerical methodology. A "preferred" choice: high-resolution shock-capturing schemes written in conservation form. (see Lecture 2)

Fluid dynamics

The defining property of fluids (liquids and gases) lies in the ease with which they may be deformed.

A "simple fluid" may be defined as a material such that the relative positions of its constituent elements change by a large amount when suitable forces, however small in magnitude, are applied to the material.

For most simple molecules, stable equilibrium between two molecules is achieved when their separation $d_0 \sim 3\text{-}4 \times 10^{-8}$ cm.

Average spacing for gases $\sim 10\ d_0$, while in liquids and solids is $\sim d_0$.

Fluid dynamics deals with the **behaviour of matter in the large** (average quantities per unit volume), on a macroscopic scale large compared with the distance between molecules, $l \gg d_0$, neglecting the molecular structure of fluids.

Macroscopic behaviour of fluids assumed to be continuous in structure, and physical quantities such as mass, density, or momentum contained within a given small volume are regarded as uniformly spread over that volume.

The quantities that characterize a fluid (in the continuum limit) are functions of time and position:

$$\begin{aligned} \rho &: \quad (t,\vec{r}) \in \mathbb{R}^4 \longrightarrow \rho(t,\vec{r}) \in \mathbb{R} \\ \vec{v} &: \quad (t,\vec{r}) \in \mathbb{R}^4 \longrightarrow \vec{v}(t,\vec{r}) \in \mathbb{R}^3 \\ \Pi &: \quad (t,\vec{r}) \in \mathbb{R}^4 \longrightarrow \Pi(t,\vec{r}) \in \mathbb{R}^9 \end{aligned}$$

density (scalar field)

velocity (vector field)

pressure tensor (tensor field)

Eulerian description: time variation of fluid properties in a fixed position in space.

Lagrangian description: variation of properties of a “fluid particle” along its motion.

Both descriptions are equivalent.

Reynold's transport theorem:

Scalar field

$$\frac{d}{dt}\int_{V_t} f\,dV \;=\; \int_{V_t}\left[\frac{\partial f}{\partial t}+\nabla\cdot(f\vec{v})\right]dV, \qquad f=f(t,\vec{r})$$

$$\frac{d}{dt}\int_{V_t} \vec{F}\,dV \;=\; \int_{V_t}\left[\frac{\partial \vec{F}}{\partial t}+(\vec{v}\cdot\nabla)\vec{F}+\vec{F}(\nabla\cdot\vec{v})\right]dV, \quad \vec{F}=\vec{F}(t,\vec{r})$$

Vector field

V_t is a volume which moves with the fluid (Lagrangian description; image of V_0 by the diffeomorphism given by the flux function).

Mass conservation (continuity equation)

Let V_t be a volume which moves with the fluid; its mass is given by:

$$m(V_t) = \int_{V_t} \rho(t, \vec{r})\, dV$$

Principle of conservation of mass enclosed within V_t:

$$\frac{d}{dt} m(V_t) = \frac{d}{dt} \int_{V_t} \rho(t, \vec{r})\, dV = 0$$

Applying the transport theorem for the density (scalar field):

$$0 = \frac{d}{dt} \int_{V_t} \rho(t, \vec{r})\, dV = \int_{V_t} \left[\frac{D\rho}{Dt} + \rho(\nabla \cdot \vec{v}) \right] dV$$

where the convective derivative is defined as $\dfrac{D}{Dt} \equiv \dfrac{\partial}{\partial t} + \vec{v} \cdot \nabla$

Since the previous equation must hold for any volume V_t we obtain the continuity equation:

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \vec{v}) = 0 \Rightarrow \frac{D \log \rho}{Dt} = -\Theta \Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0}$$

Corolary:

$$-\frac{\partial}{\partial t} \int_V \rho \, dV = \int_{\partial V} \rho \vec{v} \cdot d\vec{\Sigma}$$

the variation of the mass enclosed in a fixed volume V is equal to the flux of mass across the surface at the boundary of the volume.

Incompressible fluid:

$$\nabla \cdot \vec{v} = 0 \Leftrightarrow \frac{D\rho}{Dt} = 0$$

Momentum balance (Euler's equation)

"variation of momentum of a portion of fluid equals the net force (stresses plus external forces) exerted on it" (Newton's 2nd law):

$$\frac{d}{dt}\int_{V_t}\rho\vec{v}\,dV = -\int_{\partial V_t} p\,d\vec{\Sigma} + \int_{V_t}\vec{G}\,dV = \int_{V_t}[\vec{G}-\nabla p]\,dV$$

Applying transport theorem on the l.h.s. of above eq.:

$$\int_{V_t}\left[\frac{\partial}{\partial t}(\rho\vec{v}) + (\vec{v}\cdot\nabla)(\rho\vec{v}) + \rho\vec{v}(\nabla\cdot\vec{v})\right]dV = \int_{V_t}[\vec{G}-\nabla p]\,dV$$

which must be valid for any volume V_t, hence:

$$\frac{\partial}{\partial t}(\rho\vec{v}) + (\vec{v}\cdot\nabla)(\rho\vec{v}) + \rho\vec{v}(\nabla\cdot\vec{v}) = \vec{G}-\nabla p$$

After some algebra and using the continuity eq. we obtain Euler's eq.:

$$\boxed{\rho\frac{D\vec{v}}{Dt} = \vec{G}-\nabla p \Leftrightarrow \rho\vec{a} = \vec{G}-\nabla p}$$

Energy conservation

Let E be the total (kinetic + internal) energy of the fluid:

$$E = E_{\rm K} + E_{\rm int} = \frac{1}{2}\int_{V_t} \rho\vec{v}^2\, dV + \int_{V_t} \rho\varepsilon\, dV$$

Principle of energy conservation: "the variation in time of the total energy of a portion of fluid equals the work done per unit time by the stresses (internal forces) and the external forces".

$$\frac{dE}{dt} = \frac{d}{dt}\int_{V_t}\left(\frac{1}{2}\rho\vec{v}^2 + \rho\varepsilon\right)\, dV = -\int_{\partial V_t} p\vec{v}\cdot d\vec{\Sigma} + \int_{V_t}\vec{G}\cdot\vec{v}\, dV$$

After some algebra (transport theorem, divergence theorem):

$$\int_{V_t}\left(\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\vec{v}^2 + \rho\varepsilon\right) + \nabla\cdot\left[\left(\frac{1}{2}\rho\vec{v}^2 + \rho\varepsilon + p\right)\vec{v}\right]\right)\, dV = \int_{V_t}\rho\vec{g}\cdot\vec{v}\, dV \qquad \vec{g} = \frac{\vec{G}}{\rho}$$

which implies:

$$\boxed{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\vec{v}^2 + \rho\varepsilon\right) + \nabla\cdot\left[\left(\frac{1}{2}\rho\vec{v}^2 + \rho\varepsilon + p\right)\vec{v}\right] = \rho\vec{g}\cdot\vec{v}}$$

Hyperbolic system of conservation laws

The equations of perfect fluid dynamics are a nonlinear hyperbolic system of conservation laws:

$$\boxed{\frac{\partial \vec{u}}{\partial t} + \frac{\partial \vec{f^i}}{\partial x^i} = \vec{s}(\vec{u})}$$

$$\vec{u} = (\rho, \rho\, v^j, e) \qquad \text{(state vector)}$$

$$\vec{f^i} = (\rho\, v^i, \rho\, v^i\, v^j + p\, \delta^{ij}, (e+p)\, v^i) \qquad \text{(fluxes)}$$

$$\vec{s} = \left(0, -\rho\, \frac{\partial \Phi}{\partial x^j} + Q^j_M, -\rho\, v^i\, \frac{\partial \Phi}{\partial x^i} + Q_E + v^i\, Q^i_M\right)$$

(sources)

$\vec{g}$ is a conservative external force field (e.g. gravitational field):

$$\vec{g} = -\nabla \Phi \qquad \Delta \Phi = 4\pi G \rho$$

Q_M^i, Q_E are source terms in the momentum and energy equations, respectively, due to coupling between matter and radiation (when transport phenomena are also taken into account).

Hyperbolic equations have finite propagation speed: information can travel with limited speed, at most that given by the largest characteristic curves of the system.

The range of influence of the solution is bounded by the eigenvalues of the Jacobian matrix of the system.

$$A = \frac{\partial \vec{f^i}}{\partial \vec{u}} \Rightarrow \lambda_0 = v_i, \; \lambda_\pm = v_i \pm c_s$$

(link with numerical schemes in Lecture 2)

A bit on viscous fluids

A perfect fluid definition: force across the surface separating two fluid particles is normal to that surface.

Kinetic theory: the existence of velocity gradients implies the appearance of a force tangent to the surface separating two fluid layers (across which there is molecular difussion).

$$d\vec{F} = -p\, d\vec{\Sigma} \;\Rightarrow\; d\vec{F} = -\Pi\, d\vec{\Sigma} \qquad \Pi = p\mathbf{I} - \mathcal{S}$$

Π pressure tensor

$\mathcal{S}$ stress tensor

$$\mathcal{S} = \underbrace{2\mu}_{\text{shear and bulk viscosities}} \underbrace{\left(D - \frac{1}{3}\Theta\mathbb{I} \right)}_{\text{distortion}} + \xi\Theta\mathbb{I}$$

expansion

Using pressure tensor in the previous derivation of the Euler and energy eqs. yields viscous versions:

$$\rho\frac{D\vec{v}}{Dt} = \vec{G} - \nabla p + \mu\Delta\vec{v} + \left(\xi + \frac{1}{3}\mu\right)\nabla\cdot(\nabla\cdot\vec{v})$$

Navier-Stokes eq.

$$\rho\frac{D\left(\frac{1}{2}\vec{v}^2 + \varepsilon\right)}{Dt} = \rho\vec{g}\cdot\vec{v} - \nabla\cdot(p\vec{v}) + \nabla\cdot(\mathcal{S}\cdot\vec{v}) - \nabla\cdot\vec{Q}$$

Energy eq.

General relativistic hydrodynamics

The general relativistic hydrodynamics equations are obtained from the local conservation laws of the stress-energy tensor, $T^{\mu\nu}$ (the Bianchi identities), and of the matter current density J^{μ} (the continuity equation):

$$\nabla_\mu (\rho u^\mu) = 0 \qquad \nabla_\mu T^{\mu\nu} = 0$$

Equations of motion

$(\mu = 0, \cdots, 3)$

∇_μ	covariant derivative associated with 4-dimensional spacetime metric $g_{\mu\nu}$
$J^\mu = \rho u^\mu$	current density
u^μ	fluid 4-velocity
ρ	rest-mass (baryon) density in a locally inertial reference frame

Stress-energy tensor for a non-perfect fluid defined as:

$$T^{\mu\nu} = \rho(1+\varepsilon)u^\mu u^\nu + (p - \mu\Theta)h^{\mu\nu} - 2\xi\sigma^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu$$

ε specific internal energy density

$h^{\mu\nu} = u^\mu u^\nu + g^{\mu\nu}$ spatial projection tensor

$\Theta = \nabla_\mu u^\mu$ expansion [describes divergence/ convergence of fluid world lines]

$$\sigma^{\mu\nu} = \frac{1}{2}(\nabla_\alpha u^\mu h^{\alpha\nu} + \nabla_\alpha u^\nu h^{\alpha\mu}) - \frac{1}{3}\Theta h^{\mu\nu}$$

spatial shear tensor

q^μ energy flux vector

In the following we will neglect non-adiabatic effects, such as viscosity or heat transfer, assuming the stress-energy tensor to be that of a perfect fluid:

$$T^{\mu\nu} = \rho h u^\mu u^\nu + p g^{\mu\nu}$$

$$h = 1 + \varepsilon + \frac{p}{\rho}$$ relativistic specific enthalpy

Conservation laws w.r.t. an explicit coordinate chart

$x^\mu = (x^0, x^i)$

$$\boxed{\begin{aligned} \frac{\partial}{\partial x^\mu}(\sqrt{-g}\rho u^\mu) &= 0 \\ \frac{\partial}{\partial x^\mu}(\sqrt{-g}T^{\mu\nu}) &= \sqrt{-g}\Gamma^\nu_{\mu\lambda}T^{\mu\lambda} \end{aligned}}$$

$g = \det g_{\mu\nu}$ $\Gamma^\nu_{\mu\lambda}$ Christoffel symbols

System formed by eqs of motion and continuity eq must be supplemented with an equation of state (EOS) relating the pressure to some fundamental thermodynamical quantities,

$$p = p(\rho, \varepsilon)$$

Perfect fluid: $p = (\Gamma - 1)\rho\varepsilon$

Polytrope: $p = \kappa\rho^{\Gamma}, \quad \Gamma = 1 + \frac{1}{N}$

In the "test-fluid" approximation (fluid's self-gravity neglected) dynamics of matter fields fully described by the previous conservation laws and the EOS.

Otherwise, equations must be solved in conjunction with Einstein's equations for the gravitational field which describe the evolution of a dynamical spacetime:

$$\nabla_\mu(\rho u^\mu) = 0$$

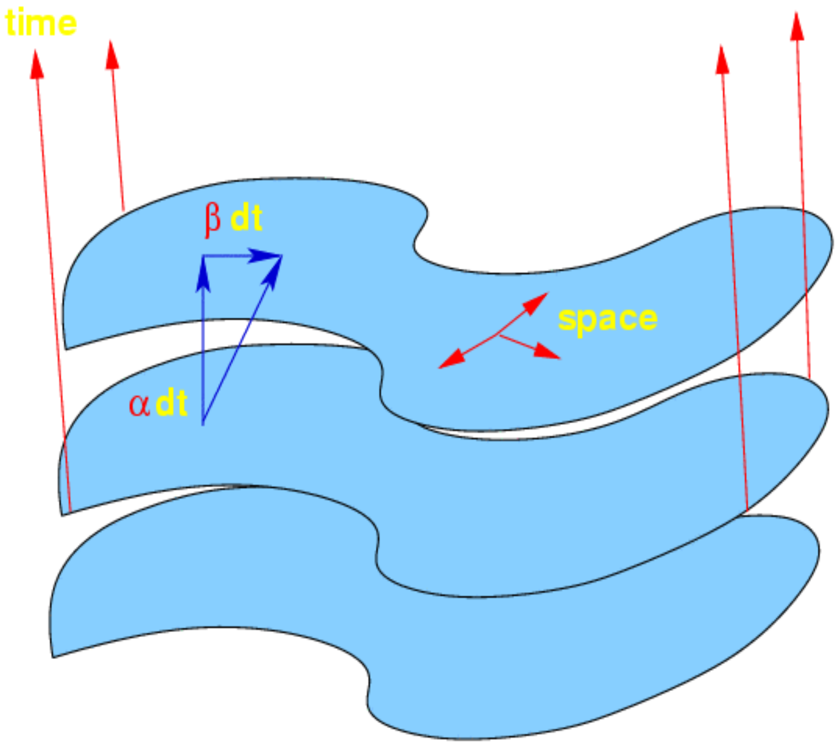
$$\nabla_\mu T^{\mu\nu} = 0$$

$$p = p(\rho, \varepsilon)$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}$$

Einstein's equations

(Newtonian analogy: Euler's equation + Poisson's equation)



The most widely used approach to solve Einstein's equations in Numerical Relativity is the 3+1 formulation.

Spacetime is foliated with a set of non-intersecting spacelike hypersurfaces Σ. Within each surface distances are measured with the spatial 3-metric.

Two kinematical variables describe the evolution between each hypersurface: the lapse function, and the shift vector.

$$ds^2 = -(\alpha^2 - \beta_i \beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j$$

3+1 GR Hydro equations: formulations

$$
\begin{aligned}
\frac{\partial}{\partial x^\mu}(\sqrt{-g}\rho u^\mu) &= 0 \\
\frac{\partial}{\partial x^\mu}(\sqrt{-g}T^{\mu\nu}) &= \sqrt{-g}\Gamma^\nu_{\mu\lambda}T^{\mu\lambda}
\end{aligned}
$$

Different formulations exist depending on:

1. Choice of slicing: level surfaces of x^0 spatial (3+1) or null
2. Choice of physical (primitive) variables (ρ, ε, u^i …)

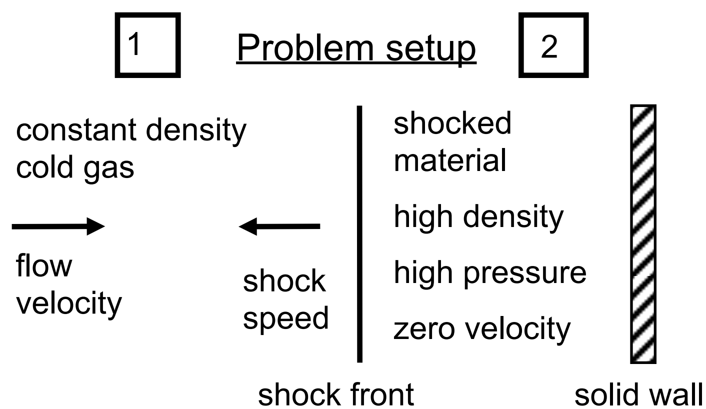
Wilson (1972) wrote the system as a set of advection equation within the 3+1 formalism. Non-conservative.

Conservative formulations well-adapted to numerical methodology:

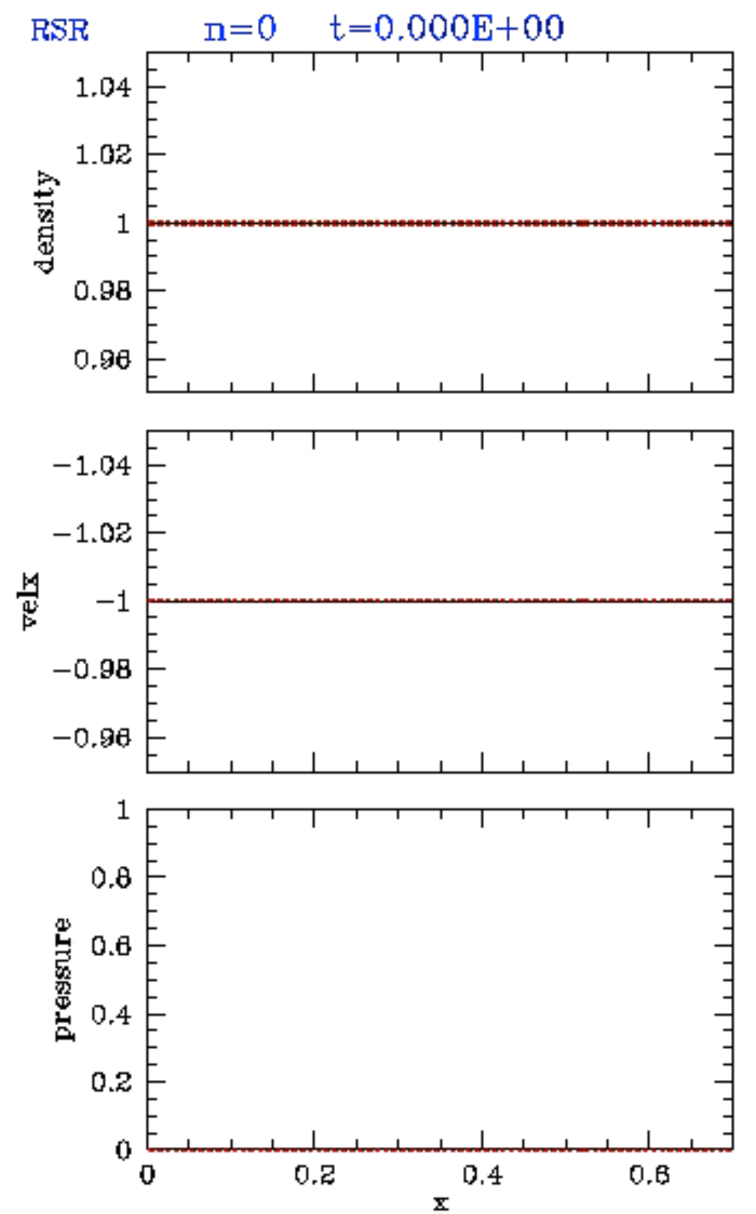
- Martí, Ibáñez & Miralles (1991): 1+1, general EOS
- Eulderink & Mellema (1995): covariant, perfect fluid
- Banyuls et al (1997): 3+1, general EOS
- Papadopoulos & Font (2000): covariant, general EOS

Relativistic shock reflection

The relativistic shock reflection problem is a demanding test involving the heating of a cold gas which impacts at relativistic speed with a solid wall creating a shock which propagates off the wall.



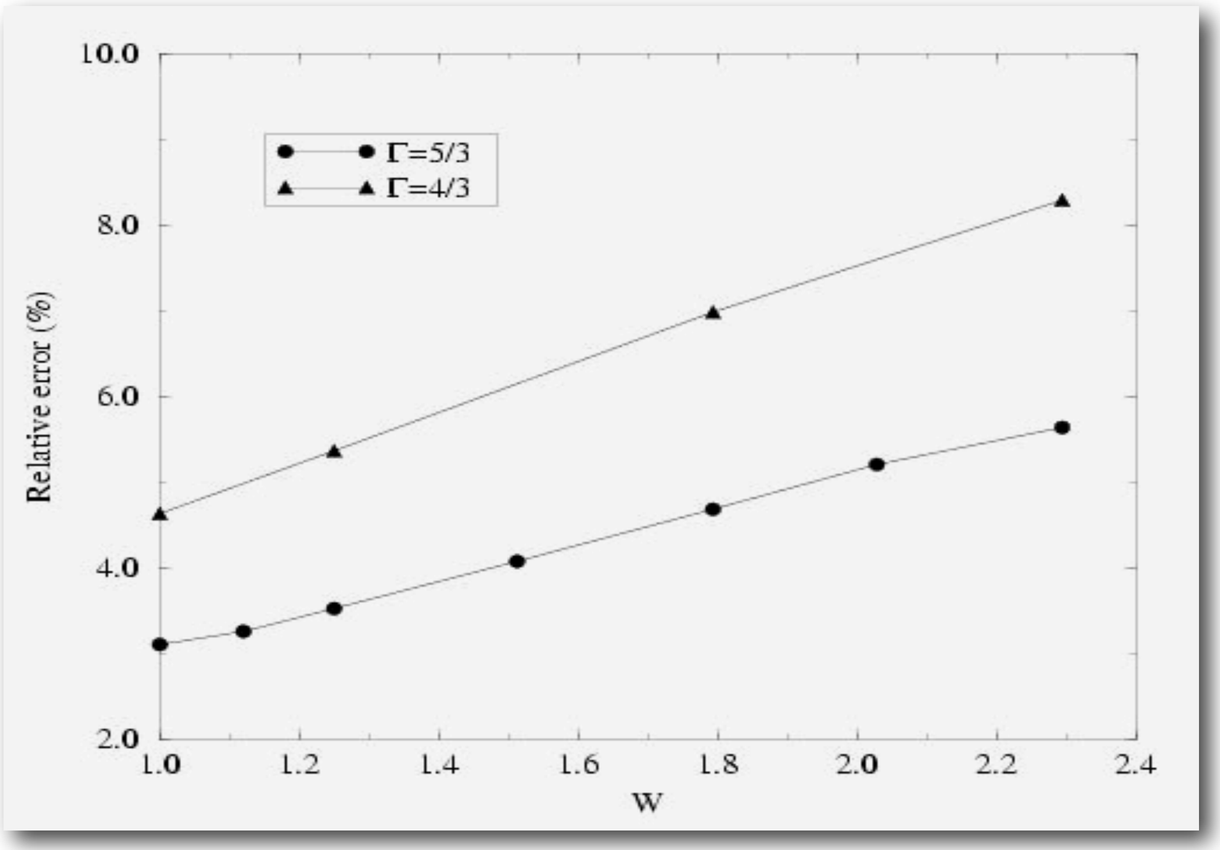
Analytic solution: $\frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1} + \frac{\gamma}{\gamma-1}\varepsilon_2, \quad \varepsilon_2 = W_1 - 1$



(from Martí & Müller, 2003)

Non-conservative formulations show limitations when simulating ultrarelativistic flows (Centrella & Wilson 1984, Norman & Winkler 1986).

Relativistic shock reflection test relative errors as a function of the fluid's Lorentz factor W. For W≈2 (v≈0.86c), error ~ 5-7% (depends on the adiabatic index of the EOS) and shows a linear increase with W.



Ultrarelativistic flows could only be handled once conservative formulations were adopted (Martí, Ibáñez & Miralles 1991; Marquina et al 1992)

Valencia's conservative formulation (Banyuls et al 1997)

Numerically, the hyperbolic and conservative nature of the GRHD equations allows to design a solution procedure based on the characteristic speeds and fields of the system, translating to relativistic hydrodynamics existing tools of CFD.

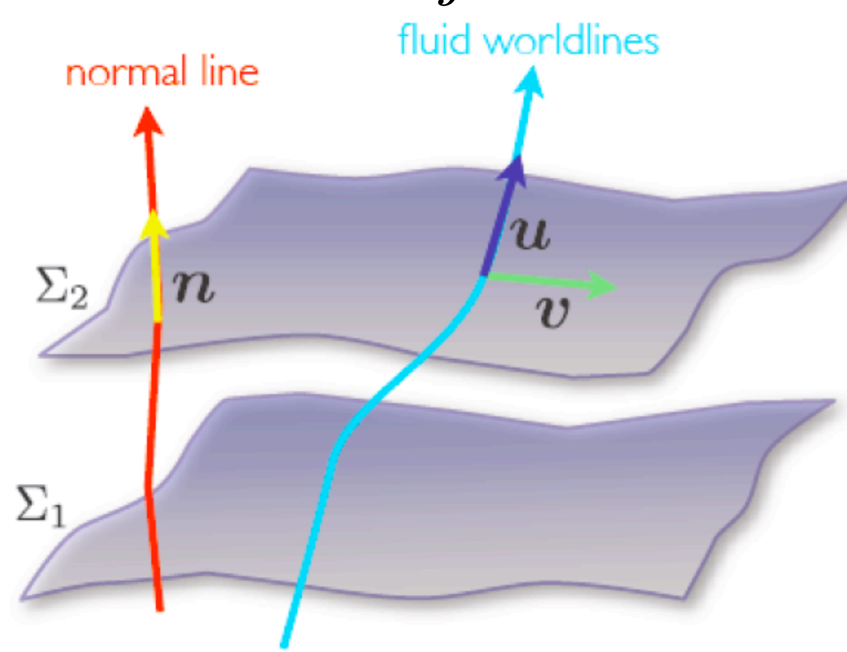
3+1: foliation of spacetime with spatial hypersurfaces Σ_t with constant t.

Line element:

$$ds^2 = -(\alpha^2 - \beta_i \beta^i) dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j$$

Eulerian observer: at rest on the hypersurface; moves from Σ_t to $\Sigma_{t+\Delta t}$ along the unit normal vector. Speed given by:

$$v^i = \frac{1}{\alpha}\left(\frac{u^i}{u^t} + \beta^i\right)$$



Hyperbolic system:

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{\gamma}\mathbf{U}}{\partial x^0}+\frac{\partial\sqrt{-g}\mathbf{F}^i}{\partial x^i}\right)=\mathbf{S}$$

$$\begin{aligned}
\mathbf{U} &= (D, S_j, \tau) \\
\mathbf{F}^i &= \left(D\left(v^i-\frac{\beta^i}{\alpha}\right), S_j\left(v^i-\frac{\beta^i}{\alpha}\right)+p\delta^i_j, \tau\left(v^i-\frac{\beta^i}{\alpha}\right)+pv^i\right) \\
\mathbf{S} &= \left(0, T^{\mu\nu}\left(\frac{\partial g_{\nu j}}{\partial x^\mu}-\Gamma^\delta_{\nu\mu}g_{\delta j}\right), \alpha\left(T^{\mu 0}\frac{\partial \ln\alpha}{\partial x^\mu}-T^{\mu\nu}\Gamma^0_{\nu\mu}\right)\right)
\end{aligned}$$

First-order flux-conservative hyperbolic system

$$\begin{aligned}
D &= \rho W \\
S_j &= \rho h W^2 v_j \\
\tau &= \rho h W^2 - p - D
\end{aligned}$$

$$W^2=\frac{1}{1-v^j v_j}$$

Lorentz factor

$$h=1+\varepsilon+\frac{p}{\rho}$$

specific enthalpy

Recovering special relativistic and Newtonian limits

Full GR

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{\gamma}\rho W}{\partial t}+\frac{\partial\sqrt{-g}\rho W v^i}{\partial x^i}\right)=0$$

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{\gamma}\rho h W^2 v^j}{\partial t}+\frac{\partial\sqrt{-g}(\rho h W^2 v^i v^j+p\delta^{ij})}{\partial x^i}\right)=T^{\mu\nu}\left(\frac{\partial g_{\nu j}}{\partial x^\mu}-\Gamma^\delta_{\nu\mu}g_{\delta j}\right)$$

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{\gamma}(\rho h W^2-p-\rho W)}{\partial t}+\frac{\partial\sqrt{-g}(\rho h W^2-\rho W)v^i}{\partial x^i}\right)=\alpha\left(T^{\mu 0}\frac{\partial \ln\alpha}{\partial x^\mu}-T^{\mu\nu}\Gamma^0_{\nu\mu}\right)$$

$$\Downarrow$$

Minkowski

$$\frac{\partial\rho W}{\partial t}+\frac{\partial\rho W v^i}{\partial x^i}=0$$

$$\frac{\partial\rho h W^2 v^j}{\partial t}+\frac{\partial(\rho h W^2 v^i v^j+p\delta^{ij})}{\partial x^i}=0$$

$$\frac{\partial(\rho h W^2-p-\rho W)}{\partial t}+\frac{\partial(\rho h W^2-\rho W)v^i}{\partial x^i}=0$$

Newton

$$\frac{\partial\rho}{\partial t}+\frac{\partial\rho v^i}{\partial x^i}=0$$

$$\frac{\partial\rho v^j}{\partial t}+\frac{\partial(\rho v^i v^j+p\delta^{ij})}{\partial x^i}=0$$

$$\frac{\partial\left(\rho\varepsilon+\frac{1}{2}\rho v^2\right)}{\partial t}+\frac{\partial\left(\rho\varepsilon+\frac{1}{2}\rho v^2+p\right)v^i}{\partial x^i}=0$$

(quiz: prove it!)

Recovering the primitive variables

A distinctive feature of the numerical solution of the relativistic HD equations is that while the numerical algorithm updates the vector of conserved quantities, the numerical code makes extensive use of the primitive variables.

$$\mathbf{U} = (D, S_i, \tau) \Leftrightarrow \mathbf{w} = (\rho, v_i, \varepsilon)$$

Those would appear in the solution procedure, e.g. in the characteristic fields, in the solution of the Riemann problem, and in the computation of the numerical fluxes.

In 3+1 NR the relation between two sets of variables is implicit. Hence, iterative (root-finding) algorithms are required.

This feature may lead to accuracy losses in regions of low density and small velocities, apart from being computationally inefficient.

Example of primitive recovery

Newtonian hydro: explicit to obtain "primitive" variables from state vector.

3+1 GR hydro: root-finding procedure. The expressions relating the primitive variables to the state vector depend explicitly on the EOS. Simple expressions are only obtained for simple EOS, i.e. ideal gas.

One can build a function of pressure whose zero represents the pressure in the physical state (other choices possible):

$$\vec{w} = (\rho, \varepsilon, v^i) \Rightarrow \begin{aligned} D &= \rho W \\ S_j &= \rho h W^2 v_j \\ \tau &= \rho h W^2 - p - D \end{aligned}$$

$$\boxed{f(\bar{p}) = p(\rho_*(\bar{p}), \varepsilon_*(\bar{p})) - \bar{p}}$$

$$\rho_*(\bar{p}) = \frac{D}{W_*(\bar{p})} \qquad \varepsilon_*(\bar{p}) = \frac{\tau + D[1 - W_*(\bar{p})] + \bar{p}[1 - W_*(\bar{p})^2]}{D W_*(\bar{p})}$$

$$W_*(\bar{p}) = \frac{1}{\sqrt{1 - v_*^i(\bar{p})\, v_{*i}(\bar{p})}} \qquad v_*^i(\bar{p}) = \frac{S^i}{\tau + D + \bar{p}}$$

The root of the above function can be obtained by means of a nonlinear root-finder (e.g. Newton-Raphson method).

Eigenvalues (characteristic speeds)

Numerical schemes based on Riemann solvers use the local characteristic structure of the hyperbolic system of equations.

The eigenvalues (characteristic speeds) are all real (but not distinct, one showing a threefold degeneracy), and a complete set of right-eigenvectors exists. The above system satisfies, hence, the definition of hyperbolicity.

Eigenvalues (along the x direction)

$$\boxed{\begin{aligned} \lambda_0 &= \alpha v^x - \beta^x \quad \text{(triple)} \\ \lambda_\pm &= \frac{\alpha}{1 - v^2 c_s^2}\left\{ v^x (1 - c_s^2) \pm c_s \sqrt{(1 - v^2)[\gamma^{xx}(1 - v^2 c_s^2) - v^x v^x (1 - c_s^2)]} \right\} - \beta^x \end{aligned}}$$

Eigenvalues (along the x direction)

$$\lambda_0 = \alpha v^x - \beta^x \quad \text{(triple)}$$

$$\lambda_\pm = \frac{\alpha}{1 - v^2 c_s^2}\left\{v^x(1-c_s^2) \pm c_s \sqrt{(1-v^2)[\gamma^{xx}(1-v^2c_s^2) - v^x v^x(1-c_s^2)]}\right\} - \beta^x$$

Special relativistic limit (along x-direction)

$$\lambda_0 = v^x$$

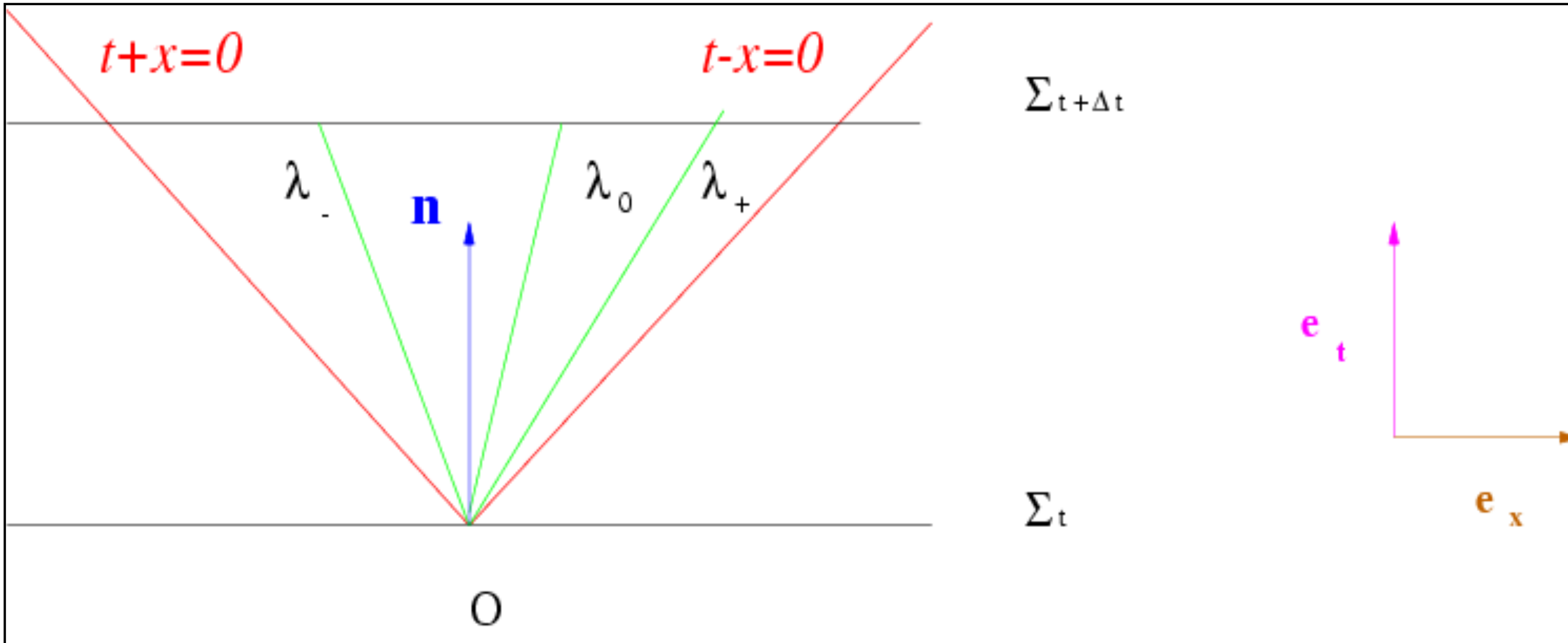
$$\lambda_\pm = \frac{1}{1 - v^2 c_s^2}[v^x(1-c_s^2) \pm c_s \sqrt{(1-v^2)(1-v^2c_s^2 - v^x v^x(1-c_s^2))}]$$

coupling with transversal components of the velocity
(important difference with Newtonian case)

Even in purely 1D case: $\vec{v} = (v^x, 0, 0) \Rightarrow \lambda_0 = v^x, \; \lambda_\pm = \dfrac{v^x \pm c_s}{1 \pm v^x c_s}$

Recall Newtonian (1D) case: $\lambda_0 = v^x, \; \lambda_\pm = v^x \pm c_s$

For causal EOS sound cone lies within light cone



$$\mathbf{e}_t \cdot \mathbf{e}_t = -1\,,\quad \mathbf{e}_x \cdot \mathbf{e}_x = +1\,,\quad \mathbf{e}_t \cdot \mathbf{e}_x = 0$$

General Relativistic MHD (Antón et al. 2006)

GRMHD: Dynamics of relativistic, electrically conducting fluids in the presence of magnetic fields.

Ideal GRMHD: Absence of viscosity effects and heat conduction in the limit of infinite conductivity (perfect conductor fluid).

The stress-energy tensor includes contribution from the perfect fluid and from the magnetic field b^μ measured by observer comoving with the fluid.

$$T^{\mu\nu} = T^{\mu\nu}_{\mathrm{PF}} + T^{\mu\nu}_{\mathrm{EM}} \longrightarrow \boxed{T^{\mu\nu} = \rho h^* u^\mu u^\nu + p^* g^{\mu\nu} - b^\mu b^\nu}$$

$$T^{\mu\nu}_{\mathrm{PF}} = \rho h u^\mu u^\nu + p g^{\mu\nu}$$

$$T^{\mu\nu}_{\mathrm{EM}} = F^{\mu\lambda} F^{\nu}_{\lambda} - \frac{1}{4} g^{\mu\nu} F^{\lambda\delta} F_{\lambda\delta} = \left(u^\mu u^\nu + \frac{1}{2} g^{\mu\nu} \right) b^2 - b^\mu b^\nu$$

$$F^{\mu\nu} = -\eta^{\mu\nu\lambda\delta} u_\lambda b_\delta$$

with the definitions:

$$\begin{aligned} b^2 &= b^\nu b_\nu \\ p^* &= p + \frac{b^2}{2} \\ h^* &= h + \frac{b^2}{\rho} \end{aligned}$$

Ideal MHD condition: electric four-current must be finite.

$$F^{\mu\nu} u_\nu = 0$$

$$J^\mu = \rho_{\mathrm{q}} u^\mu + \sigma F^{\mu\nu} u_\nu \quad \sigma \longrightarrow \infty$$

Conservation of mass: $\nabla_\mu(\rho u^\mu) = 0$

Conservation of energy and momentum: $\nabla_\mu T^{\mu\nu} = 0$

Maxwell's equations: $\nabla_\mu {}^*F^{\mu\nu} = 0 \quad {}^*F^{\mu\nu} = \frac{1}{W}(u^\mu B^\nu - u^\nu B^\mu)$

- Divergence-free constraint: $\vec{\nabla} \cdot \vec{B} = 0$
- Induction equation: $\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial t}\left(\sqrt{\gamma}\vec{B}\right) = \vec{\nabla} \times \left[\left(\alpha\vec{v} - \vec{\beta}\right) \times \vec{B}\right]$

Adding all up:
first-order, flux-conservative, hyperbolic system + constraint

$$\boxed{\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{\gamma}\mathbf{U}}{\partial t} + \frac{\partial\sqrt{-g}\mathbf{F}^i}{\partial x^i}\right) = \mathbf{S} \qquad \frac{\partial(\sqrt{\gamma}B^i)}{\partial x^i} = 0}$$

Antón et al. (2006)

$$D = \rho W \qquad S_j = \rho h^* W^2 v_j - \alpha b_j b^0 \qquad \tau = \rho h^* W^2 - p^* - \alpha^2 (b^0)^2 - D$$

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{\gamma}\mathbf{U}}{\partial t}+\frac{\partial\sqrt{-g}\mathbf{F}^i}{\partial x^i}\right)=\mathbf{S} \qquad \frac{\partial(\sqrt{\gamma}B^i)}{\partial x^i}=0$$

$$D=\rho W \qquad S_j=\rho h^* W^2 v_j-\alpha b_j b^0 \qquad \tau=\rho h^* W^2-p^*-\alpha^2(b^0)^2-D$$

$$\mathbf{U}=\begin{bmatrix} D \\ S_j \\ \tau \\ B^k \end{bmatrix} \quad \mathbf{F}^i=\begin{bmatrix} D\tilde{v}^i \\ S_j\tilde{v}^i+p^*\delta^i_j-b_jB^i/W \\ \tau\tilde{v}^i+p^*v^i-\alpha b^0B^i/W \\ \tilde{v}^iB^k-\tilde{v}^kB^i \end{bmatrix} \quad \mathbf{S}=\begin{bmatrix} 0 \\ T^{\mu\nu}\left(\frac{\partial g_{\nu j}}{\partial x^\mu}-\Gamma^\delta_{\nu\mu}g_{\delta j}\right) \\ \alpha\left(T^{\mu 0}\frac{\partial \ln\alpha}{\partial x^\mu}-T^{\mu\nu}\Gamma^0_{\nu\mu}\right) \\ 0^k \end{bmatrix}$$

Hyperbolic and conservative nature of the GR(M)HD eqs allows to design a solution procedure based on characteristic speeds and fields of the system, translating to relativity existing tools of CFD. Godunov-type or **high-resolution shock-capturing (HRSC)** schemes.

Divergence-free constraint not guaranteed to be satisfied numerically when updating the B-field with a HRSC scheme.

Recovering primitive variables in RMHD

Significantly more involved than in RHD (see Noble et al 2006 for a comparison of methods)

Antón et al 2006 find the roots of an 8th prder polynomial using a 2D Newton-Raphson.

$$D = \rho W$$

$$S^i = \rho h^* W^2 v^i - b^i b^0$$

$$\tau = \rho h^* W^2 - D - p^* - (b^0)^2$$

$$H^i = W(b^i - v^i b^0)$$

$$\mathbf{U} = (D, S^i, \tau, H^i) \quad \rightarrow \quad (\rho, v^i, \varepsilon, B^i)$$

$$P_8 = \sum_{n=0}^{8} A_n Z^n \;, \quad Z \equiv \rho h W^2$$

Unknowns: Z and W.

$$\boxed{\begin{gathered}\left(\tau - Z - \mathbf{B}^2 + \frac{(\mathbf{BS})^2}{2Z^2}\right) W^2 + \left(\frac{\gamma-1}{\gamma}\right)(Z - DW) + \frac{\mathbf{B}^2}{2} = 0 \\[2em] \left((Z+\mathbf{B}^2)^2 - \mathbf{S}^2 - \frac{(\mathbf{BS})^2}{Z^2}(2Z+\mathbf{B}^2)\right) W^2 - (Z+\mathbf{B}^2)^2 = 0\end{gathered}}$$

MHD equations: hyperbolic structure

Wave structure classical MHD (Brio & Wu 1988): 7 physical waves

Two ALFVEN WAVES: $\lambda_{a_\pm} \Longrightarrow \lambda_a = v_x \pm v_a$

Two FAST MAGNETOSONIC WAVES: $\lambda_{f_\pm} \Longrightarrow \lambda_{f_\pm} = v_x \pm v_f$

Two SLOW MAGNETOSONIC WAVES: $\lambda_{s_\pm} \Longrightarrow \lambda_{s_\pm} = v_x \pm v_s$

One ENTROPY WAVE: $\lambda_e \Longrightarrow \lambda_e = v_x$

$$\boxed{\lambda_{f_-} \le \lambda_{a_-} \le \lambda_{s_-} \le \lambda_e \le \lambda_{s_+} \le \lambda_{a_+} \le \lambda_{f_+}}$$

$$\boxed{v_{f,s}^2 = \frac{1}{2}\left\{c_s^2 + \frac{B_x^2+B_y^2+B_z^2}{\rho} \pm \sqrt{\left(c_s^2 + \frac{B_x^2+B_y^2+B_z^2}{\rho}\right)^2 - 4\left(\frac{B_x^2}{\rho}\right)c_s^2}\right\}, \quad v_a = \sqrt{\frac{B_x^2}{\rho}}}$$

Anile & Pennisi (1987), Anile (1989) (see also van Putten 1991) have studied the characteristic structure of the equations (eigenvalues, right/left eigenvectors) in the space of covariant variables (u^μ,b^μ,p,s).

Wave structure for relativistic MHD (Anile 1989): roots of the characteristic equation.

Only entropic waves and Alfvén waves are explicit.

Magnetosonic waves are given by the numerical solution of a quartic equation.

Augmented system of equations: Unphysical eigenvalues/eigenvectors (entropy & Alfvén) which must be removed numerically (Anile 1989, Komissarov 1999, Balsara 2001, Koldoba et al 2002).

Einstein's equations and Numerical Relativity

Dynamics of gravitational field described by Einstein's field equations:

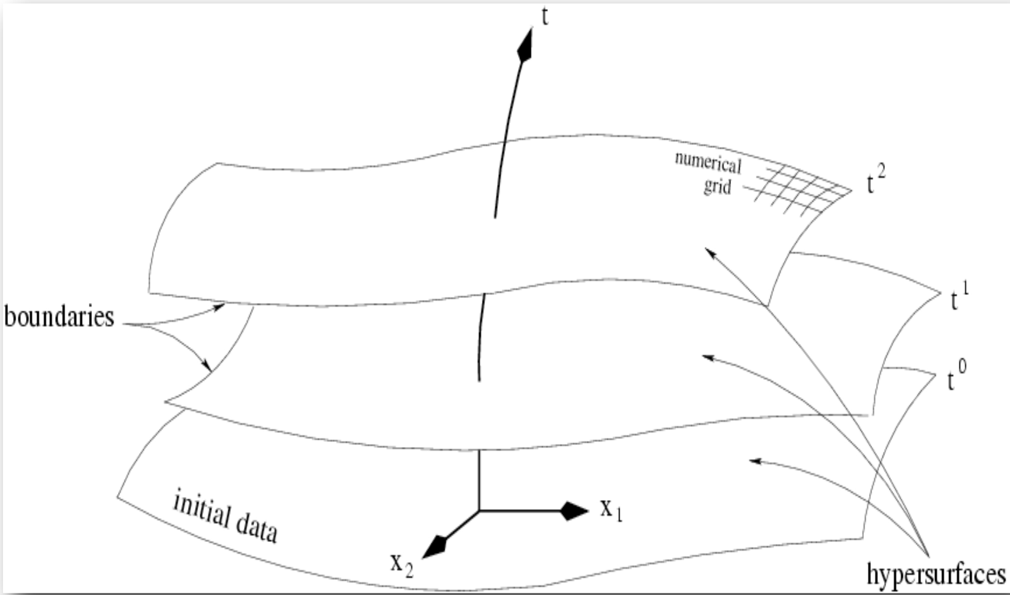
$$\boxed{G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}}$$

These equations relate the **spacetime geometry** (left-hand side) with the **distribution of matter and energy** (right-hand side): *"Matter tells spacetime how to curve, and spacetime tells matter how to move."*

Einstein's equations are a system of 10 nonlinear, coupled, partial differential equations in 4 dimensions.

When written with respect to a general coordinate system they may contain hundreds of terms ...

The most widely used approach to solve Einstein's equations in Numerical Relativity is the so-called Cauchy or 3+1 formulation (IVP).



Spacetime is foliated with a set of non-intersecting spacelike hypersurfaces Σ.

Within a given surface distances are measured with the spatial 3-metric.

Two kinematical variables describe the evolution from one hypersurface to the next: the lapse function α which describes the rate of proper time along a timelike unit vector n^μ normal to the hypersurface, and the shift vector β^i, spatial vector which describes the movement of coordinates in the hypersurface.

$$ds^2 = -(\alpha^2 - \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j$$

Intrinsic and extrinsic curvature of spatial hypersurfaces:

Intrinsic curvature given by the 3-dimensional Riemann tensor defined in terms of the 3-metric γ_{ij} .

Extrinsic curvature K_{ij} measures the change of the vector normal to the hypersurface as it is parallel-transported from one point in the hypersurface to another.

Projection operator: $P^{\alpha}_{\beta} \equiv \delta^{\alpha}_{\beta} + n^{\alpha} n_{\beta}$

Unit normal vector: $n^{\mu} = \left(\frac{1}{\alpha}, -\frac{\beta^i}{\alpha} \right), \;\; n_{\mu} = (-\alpha, 0), \;\; n^{\mu} n_{\mu} = -1$

$$K_{\alpha\beta} \equiv -P^{\mu}_{\alpha} P^{\nu}_{\beta} \nabla_{\mu} n_{\nu} = -(\nabla_{\alpha} n_{\beta} + n_{\alpha} n^{\mu} \nabla_{\mu} n_{\beta}))$$

Substituting the form of the normal vector in the definition of the extrinsic curvature, we get:

$$K_{ij} = \frac{1}{2\alpha}(-\partial_t \gamma_{ij} + \nabla_i \beta_j + \nabla_j \beta_i)$$

Einstein's equations in 3+1 form

Using the projection operator and the normal vector, Einstein's equations can be separated in **three groups:**

- Normal projection (1 equation; energy or Hamiltonian constraint)

$$n^\alpha n^\beta (G_{\alpha\beta} - 8\pi T_{\alpha\beta}) = 0$$

- Mixed projections (3 equations; momentum constraints)

$$P[n^\alpha (G_{\alpha\beta} - 8\pi T_{\alpha\beta})] = 0$$

- Projection onto the hypersurface (6 equations; evolution of the extrinsic curvature)

$$P(G_{\alpha\beta} - 8\pi T_{\alpha\beta}) = 0$$

3+1 Formulation (Cauchy)

Lichnerowicz (1944); Choquet-Bruhat (1962); Arnowitt, Deser & Misner (1962); York (1979)

Evolution equations:

$$\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i \quad (*) \\
\partial_t K_{ij} &= -\nabla_i \nabla_j \alpha + \alpha \left(R_{ij} + K\, K_{ij} - 2K_{im}K^m_j \right) + \beta^m \nabla_m K_{ij} \\
&\quad + K_{im}\nabla_j \beta^m + K_{mj}\nabla_i \beta^m - 8\pi\alpha \left(T_{ij} - \frac{1}{2}\gamma_{ij}T^m_m + \frac{1}{2}\rho\gamma_{ij} \right)
\end{aligned}$$

Constraint equations:

$$\begin{aligned}
R + K^2 - K^{ij}K_{ij} &= 16\pi\rho \\
\nabla_i \left(K^{ij} - \gamma^{ij} K \right) &= 8\pi S^j
\end{aligned}$$

(*) $(\partial_t - \mathcal{L}_\beta)\gamma_{ij} = -2\alpha K_{ij}$

Cauchy problem (IVP):

- Specify γ_{ij} , K_{ij} at t=0 subjected to the constraint equations.
- Specify coordinates through α , β^i
- Evolve the data using EE and the definition of K_{ij}

Definitions:

∇_i Covariant derivative with respect to induced 3-metric

Ricci tensor $$R_{ij} = \partial_n \Gamma^n_{ij} - \partial_j \Gamma^n_{in} + \Gamma^n_{mn}\Gamma^m_{ij} - \Gamma^n_{jm}\Gamma^m_{in}$$

Christoffel symbols $$\Gamma^i_{jk} = \frac{1}{2}\gamma^{in}\left(\frac{\partial \gamma_{nj}}{\partial x^k} + \frac{\partial \gamma_{nk}}{\partial x^j} - \frac{\partial \gamma_{jk}}{\partial x^n}\right)$$

Scalar curvature $$R = R_{ij}\gamma^{ij}$$

Trace of extrinsic curvature $$K = K_{ij}\gamma^{ij}$$

Matter fields
$$\begin{cases} \rho & \equiv & T^{\mu\nu} n_\mu n_\nu = \rho h W^2 - P \\ S^i & \equiv & - \perp^i_\mu T^{\mu\nu} n_\nu = \rho h W^2 v^i \\ S_{ij} & \equiv & \perp^\mu_i \perp^\nu_j T^{\mu\nu} = \rho h W^2 v_i v_j + \gamma_{ij} P \\ S & \equiv & \rho h W^2 v_i v^i + 3P \end{cases}$$

BSSN formulation

Kojima, Nakamura & Oohara (1987); Shibata & Nakamura (1995); Baumgarte & Shapiro (1999)

Idea: Remove mixed second derivatives in the Ricci tensor by introducing auxiliary variables. Evolution equations start to look like wave equations for 3-metric and extrinsic curvature (idea goes back to De Donder 1921; Choquet-Bruhat 1952; Fischer & Marsden 1972).

- Conformal decomposition of the 3-metric:

$$\tilde{\gamma}_{ij} = \psi^4 \gamma_{ij} \qquad\qquad \det \tilde{\gamma}_{ij} = 1$$

- BSSN evolution variables (trace of extrinsic curvature is a separate variable):

$$\begin{aligned} \phi &= \frac{1}{4} \log \psi & \qquad \tilde{\gamma}_{ij} &= e^{-4\phi} \gamma_{ij} \\ K &= \gamma^{ij} K_{ij} & \qquad \tilde{A}_{ij} &= e^{-4\phi} \left(K_{ij} - \frac{1}{3} \gamma_{ij} K \right) \end{aligned}$$

- Introduce evolution variables (gauge source functions):

$$\tilde{\Gamma}^a = \tilde{\gamma}^{ij} \tilde{\Gamma}^a_{ij} = -\partial_i \tilde{\gamma}^{ai}$$

Evolution equations in BSSN formulation

Kojima, Nakamura & Oohara (1987); Shibata & Nakamura (1995); Baumgarte & Shapiro (1999)

$$
\begin{aligned}
(\partial_t - \mathcal{L}_\beta)\tilde{\gamma}_{ij} &= -2\alpha\tilde{A}_{ij} \\
(\partial_t - \mathcal{L}_\beta)\phi &= -\frac{1}{6}\alpha K \\
(\partial_t - \mathcal{L}_\beta)K &= -\gamma^{ij}D_iD_j\alpha + \alpha\left[\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2 + \frac{1}{2}(\rho + S)\right] \\
(\partial_t - \mathcal{L}_\beta)\tilde{A}_{ij} &= e^{-4\phi}\left[-D_iD_j\alpha + \alpha(R_{ij} - S_{ij})\right]^{\mathrm{TF}} + \alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}^l_j\right) \\
(\partial_t - \mathcal{L}_\beta)\tilde{\Gamma}^i &= -2\tilde{A}^{ij}\partial_j\alpha + 2\alpha\left(\tilde{\Gamma}^i_{jk}\tilde{A}^{kj} - \frac{2}{3}\tilde{\gamma}^{ij}\partial_j K - \tilde{\gamma}^{ij}S_j + 6\tilde{A}^{ij}\partial_j\phi\right) \\
&\quad + \partial_j\left(\beta^l\tilde{\partial}_l\gamma^{ij} - 2\tilde{\gamma}^{m(j}\partial_m\beta^{i)} + \frac{2}{3}\tilde{\gamma}^{ij}\partial_l\beta^l\right)
\end{aligned}
$$

Can be solved numerically!

BSSN is currently the standard 3+1 formulation in Numerical Relativity. Long-term stable applications include strongly-gravitating systems such as neutron stars (isolated and binaries) and single and **binary black holes!**

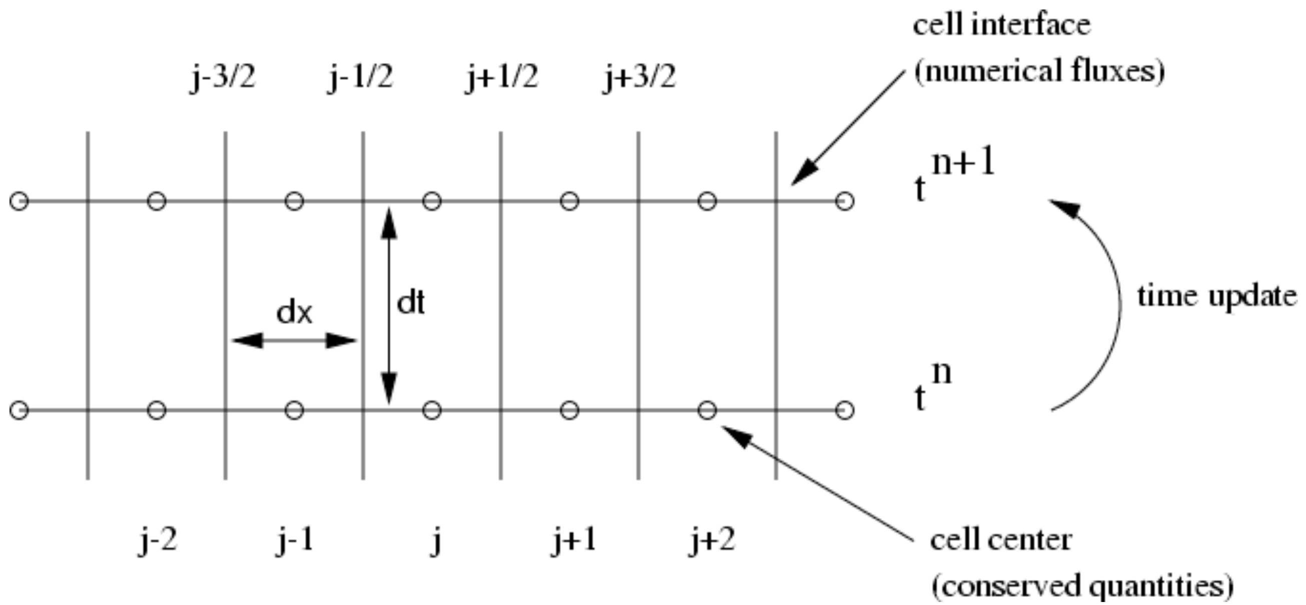
Lecture 2
Numerical methods
(Riemann solvers)

Partial differential equations (PDEs) commonly solved numerically by approximating the derivatives with difference operators. Schemes of different orders can be obtained depending on the truncation of the corresponding Taylor series for the derivatives.

Finite-difference schemes are based on a discretization of the x-t plane with a mesh of discrete points (x_j,t^n):

$$x_j = (j - 1/2)\Delta x, \quad t^n = n\Delta t, \quad j = 1, 2, \cdots \quad n = 0, 1, 2, \cdots$$

where Δx and Δt stand for the cell width and time step.



Let us consider the following scalar PDE

$$u_t + f_x = 0\,,\ \ u_0 = u(0,x)\,,\ \ f = f(u)$$

A finite-difference scheme for this eq is a time-marching procedure that permits to obtain approximations to the solution in the mesh points u_j^{n+1} from approximations in the previous time steps u_j^n

We can approximate the time derivative with a first-order forward (Euler) difference

$$u_t = \frac{u_j^{n+1} - u_j^n}{\Delta t}$$

and the spatial derivative with a first-order central difference

$$f_x = \frac{f_{j+1}^n - f_{j-1}^n}{2\Delta x}$$

which yields the explicit first-order central scheme:

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x}(f_{j+1}^n - f_{j-1}^n)$$

Many other 1st-order and higher-order approximations are available in the literature.

Method of lines (MoL)

Generic name of a family of discretization methods in which space and time variables are dealt with separately (3+1 approach). Time is discretized with finite differences and space discretization is done in various ways (finite differences, finite elements, spectral methods).

The HD, MHD, and BBSN equations can be written in a compact way as a "semi-discrete" system

$$\partial_t \mathbf{u} = \mathbf{S}$$

$\mathbf{u}$ array of dynamical fields

$\mathbf{S}$ remaining term in evolution eqs. including spatial derivatives

PDE "disguised" as an ODE (standard ODE integrators can be applied)

1st-order forward in time (Euler step)

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t\, \mathbf{S}(t^n, \mathbf{u}^n)$$

2nd-order Runge-Kutta scheme

$$\begin{aligned} \mathbf{u}^* &= \mathbf{u}^n + \Delta t\, \mathbf{S}^n \\ \mathbf{u}^{n+1} &= \frac{1}{2}\mathbf{u}^n + \frac{1}{2}\mathbf{u}^* + \frac{\Delta t}{2}\mathbf{S}^* \end{aligned}$$

higher-order in time ... many schemes available

MoL can be used with any space discretization method. FD is a fairly standard choice.

Field values at grid nodes are represented by the array

$$\mathbf{u}_{i,j,k} = \mathbf{u}(t, x_i, y_j, z_k)$$

Space derivatives:

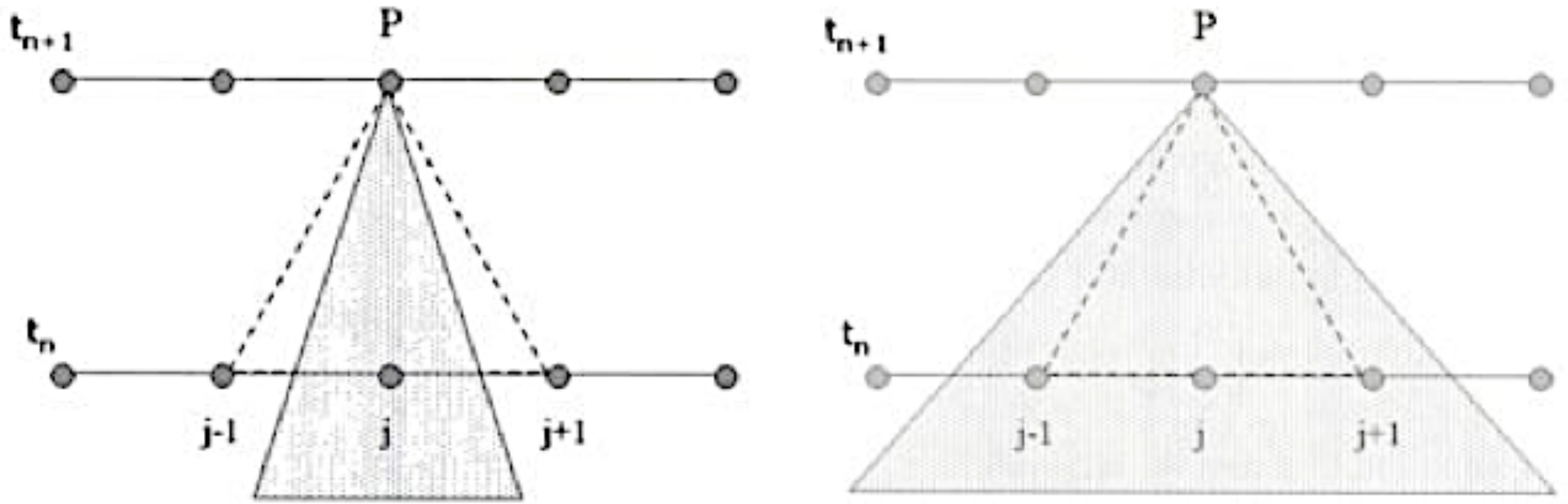
$$\begin{aligned}
2\partial_x u &\sim (u_{i+1,j,k} - u_{i-1,j,k})/\Delta x \\
2\partial_{xx} u &\sim (u_{i+1,j,k} + u_{i-1,j,k} - 2u_{i,j,k})/(\Delta x)^2 \\
2\partial_{xy} u &\sim (u_{i+1,j+1,k} - u_{i-1,j+1,k} \\
&\quad -u_{i+1,j-1,k} + u_{i-1,j-1,k})/(\Delta x \Delta y) \\
&\ldots
\end{aligned}$$

Stencil: set of grid points needed to discretize space derivatives at a given point *P*. Provides the numerical domain of dependence of selected point, i.e. any perturbation at one of the stencil points will change the computed value at *P* after a single time step.

Numerical propagation speed: $v^i_{\mathrm{num}} = s\dfrac{\Delta x^i}{\Delta t}$ s stencil size

From the physical point of view, for a system describing wave propagation with some characteristic speeds, the field values at *P* are causally determined by the values inside the past half-cone with vertex at *P*, which slope is given by the inverse of the largest characteristic speed of the system (usually the light speed, but it may also be gauge speeds).

This provides the physical domain of dependence of *P*.



Courant (necessary) condition for numerical stability: the physical domain of dependence of P must be included in the numerical domain of dependence.

$$v_{\max} < n_i \, v^i_{\mathrm{num}}$$

Provides an upper limit for the numerical time step.

The hydrodynamics equations represent processes that involve movement and wave propagation. They are hyperbolic equations. To see this we have to keep in mind that the equations can be written in the following generic 1st-order quasi-linear form:

$$\partial_t \boldsymbol{U} + \boldsymbol{A}\partial_x \boldsymbol{U} = 0 \qquad \boldsymbol{A}(\boldsymbol{U}) \equiv \partial \boldsymbol{F} / \partial \boldsymbol{U}$$

Jacobian matrix

If the Jacobian matrix has constant coefficients (linear case), the solution procedure is simple. First we diagonalize the Jacobian matrix so that

$$\Lambda = \boldsymbol{R}^{-1}\boldsymbol{A}\boldsymbol{R} \qquad \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$$

eigenvectors matrix

eigenvalues matrix

If we define the characteristic variables $\boldsymbol{W} \equiv \boldsymbol{R}^{-1}\boldsymbol{U}$

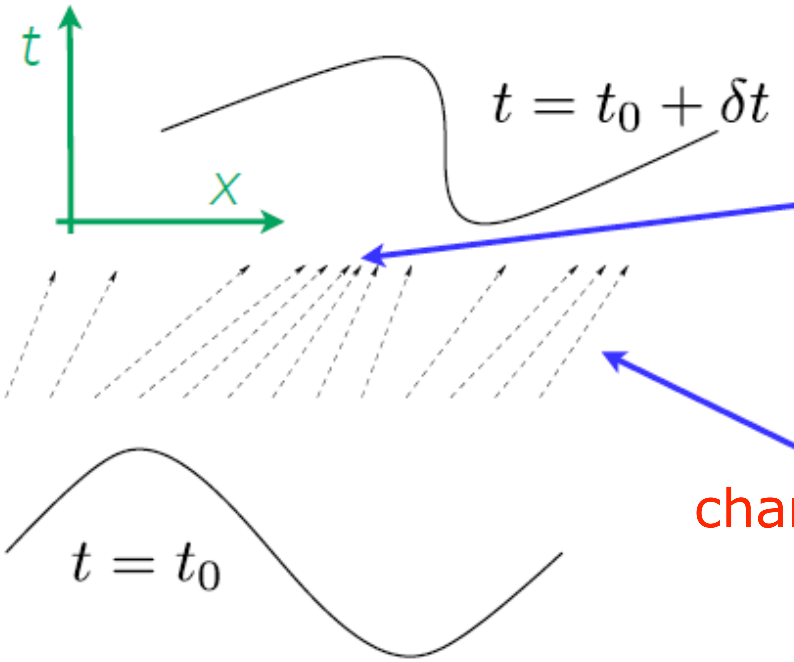
we can decouple the original system of equations:

$$\partial_t \boldsymbol{W} + \Lambda \partial_x \boldsymbol{W} = 0$$

$$\partial_t \bar{w}_i + \Lambda \partial_x \bar{w}_i = 0 \Longleftrightarrow \frac{\mathrm{d}\bar{w}_i}{\mathrm{d}t} = 0 \quad \text{along} \quad \frac{\partial x}{\partial t} = \lambda_i(\boldsymbol{U}(x,t))$$

Therefore, characteristic variables are constant along the curves of the (x,t) plane whose slopes are the corresponding eigenvalue.

Such curves are called characteristic curves and their slopes are given locally by the characteristic speeds.



the characteristic speeds are different and the characteristic curves may “focus”

characteristic curves

Since they are constant along the characteristics, the value of the characteristic variables at any given time is known once the initial value is known, that is

$$W^i(x,t) = W^i(x - \lambda_i t, t = 0)$$

Once the solution is known in terms of the characteristic variables, it is straightforward to obtain the solution in terms of the original state vector:

$$\boldsymbol{W} = \boldsymbol{R}^{-1}\boldsymbol{U} \quad \Longrightarrow \quad \boldsymbol{U} = \boldsymbol{R}\boldsymbol{W}$$

$$\boldsymbol{U}(x,t) = \sum_{i=1}^{N} W^i(x,t)\boldsymbol{R}^{(i)} = \sum_{i=1}^{N} W^i(x-\lambda_i t, 0)\boldsymbol{R}^{(i)}$$

Stated differently, the solution at any given time can be seen as the linear superposition of *N* waves, each propagating independently of the rest with a speed that is given by the corresponding eigenvalue of the Jacobian matrix of the system.

The so-called Godunov-type methods extend these concepts to nonlinear hyperbolic equations (hydrodynamics, MHD), solving Riemann problems of a new system of equations obtained by writing the original system as a quasi-linear system. The spectral information (eigen-values and vectors) of the Jacobian matrices is the basis of such solvers, as happens for linear systems.

Numerical methods in Astrophysical Fluid Dynamics

Main numerical schemes to solve the equations of a compressible fluid:

- **Finite difference methods.** Require numerical viscosity to stabilize the solution in regions where discontinuities develop.

- **Finite volume methods.** Conservation form. Use Riemann solvers to solve the equations in the presence of discontinuities **(Godunov 1959)**. **HRSC schemes.**

- **Symmetric methods.** Conservation form. Centred finite differences and high spatial order.

- **Particle methods**. Smoothed Particle Hydrodynamics (Monaghan 1992). Integrate movement of discrete particles to describe the flow. Diffusive.

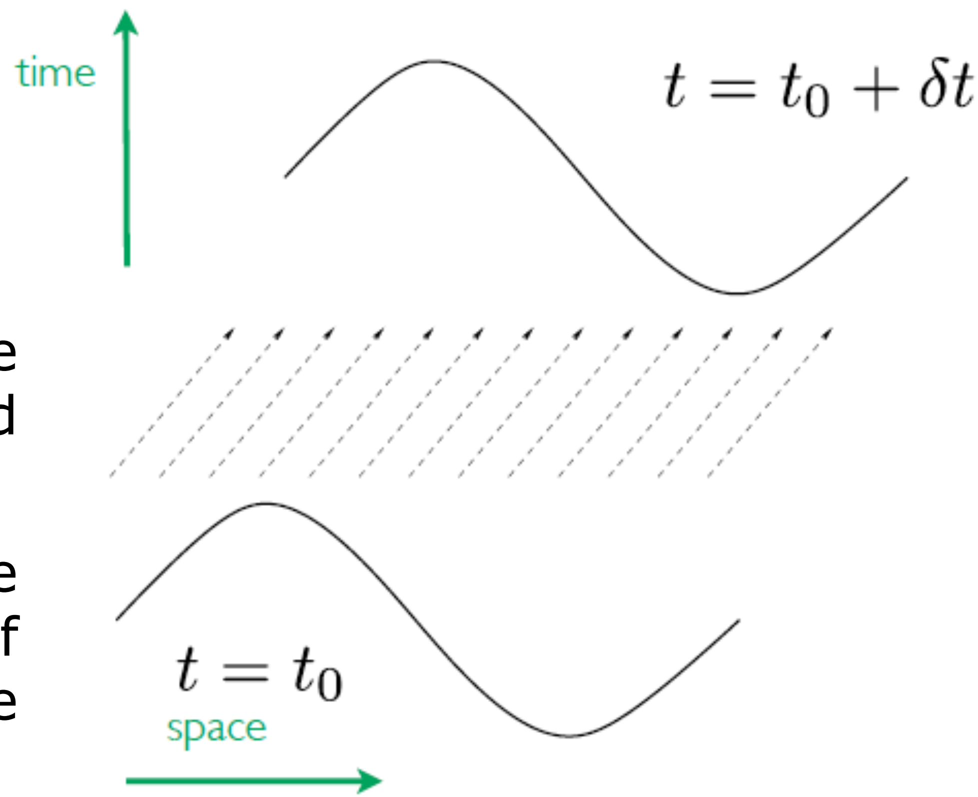
For hyperbolic systems of conservation laws, schemes written in <u>conservation form</u> guarantee that the convergence (if it exists) is to one of the weak solutions of the system of equations (Lax-Wendroff theorem 1960).

Some representative examples: advection equation

Before discussing the solution of the hydrodynamics equations there are some fundamental aspects of their nonlinear nature that must be clarified.

The simplest **linear** hyperbolic equation is the advection equation:

$$\partial_t u(x,t) + \partial_x u(x,t) = 0$$



The solution is the initial one simply translated in space and time.

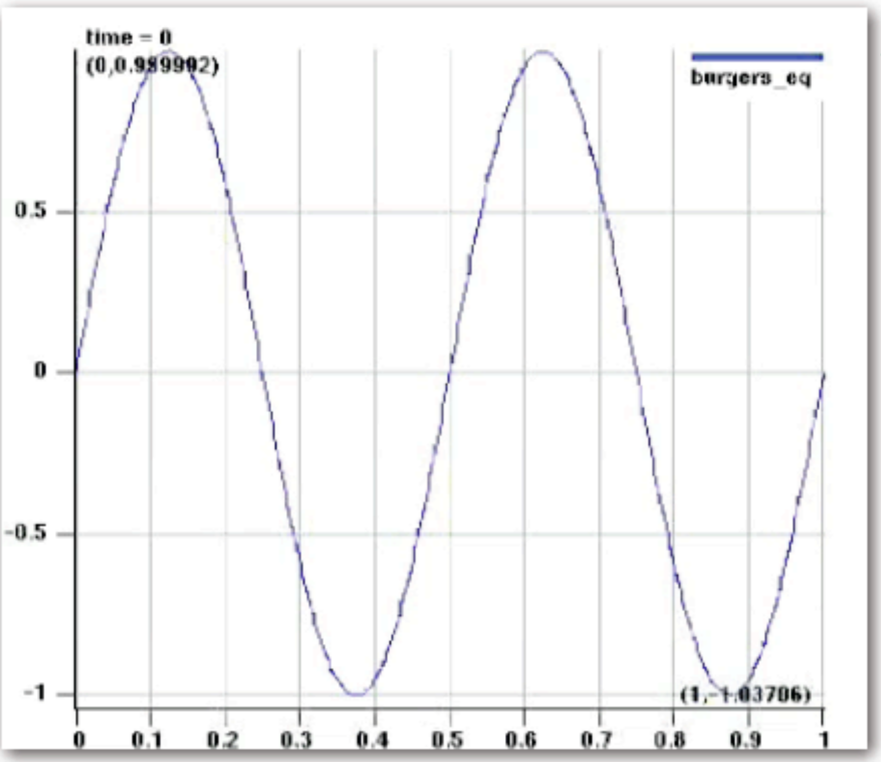
The propagation speeds are constant in every point of space (linear nature of the equation).

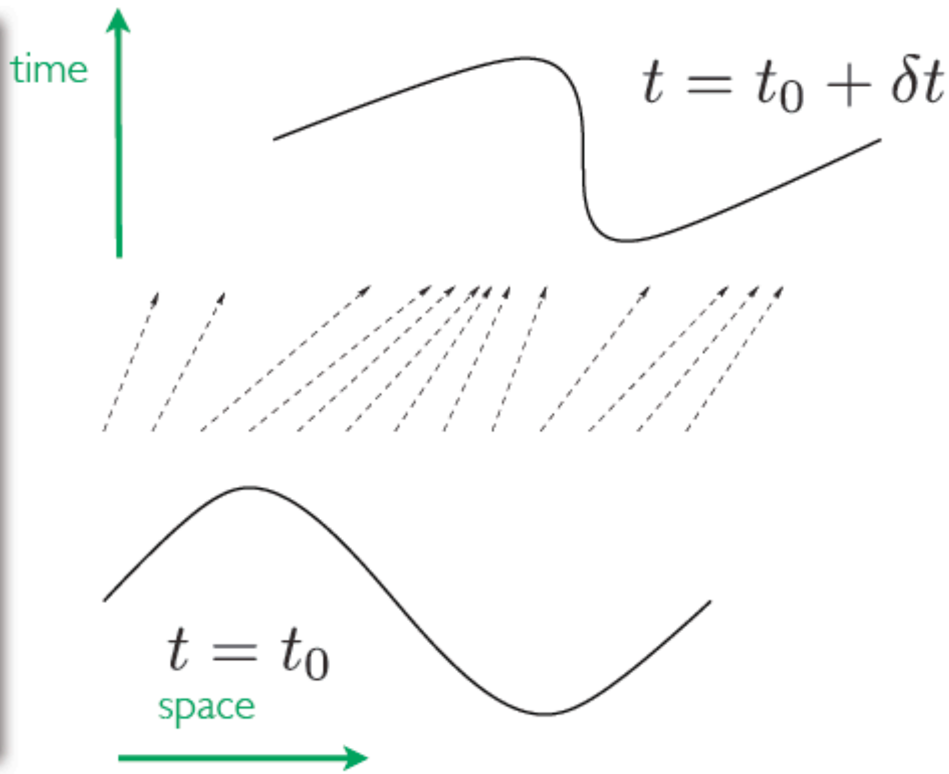
Some representative examples: Burgers equation

The simplest **nonlinear** hyperbolic equation is Burgers equation

$$\partial_t u(x,t) + u(x,t)\partial_x u(x,t) = \epsilon(x,t)\partial_x^2 u(x,t)$$

where the r.h.s. is zero in the inviscid limit. Despite the similarity with the advection equation, its solution is markedly different.





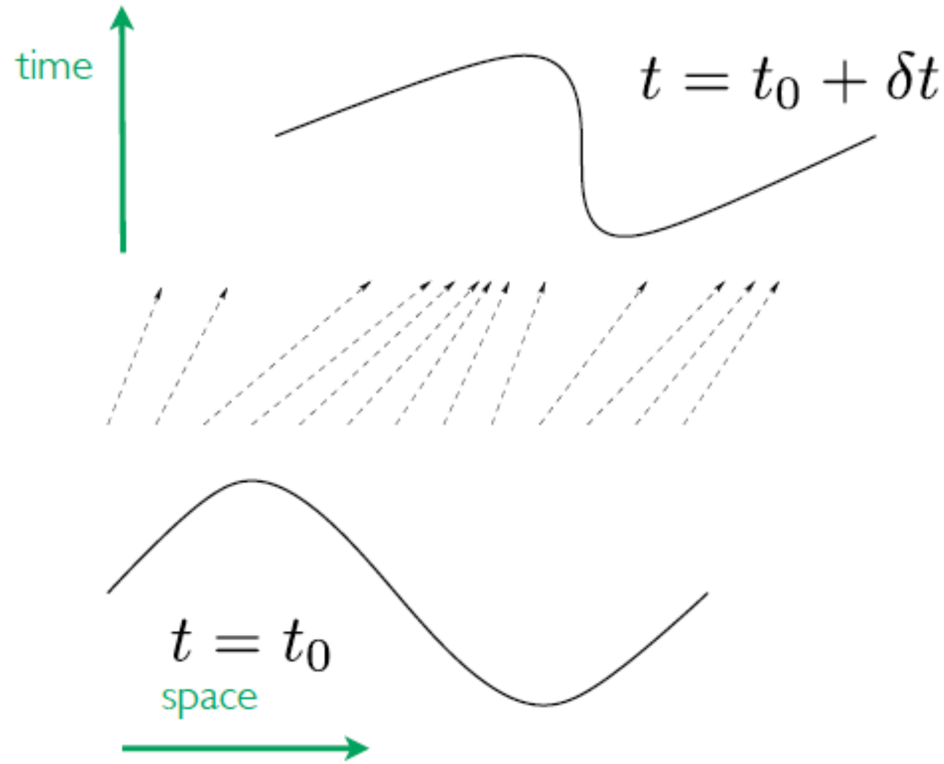
Some representative examples: Burgers equation

This behaviour is known as "shock steepening" and is a consequence of the propagation speeds not being constant, contrary to what happens with the advection equation, but are functions of space and time (nonlinear nature of the equation).

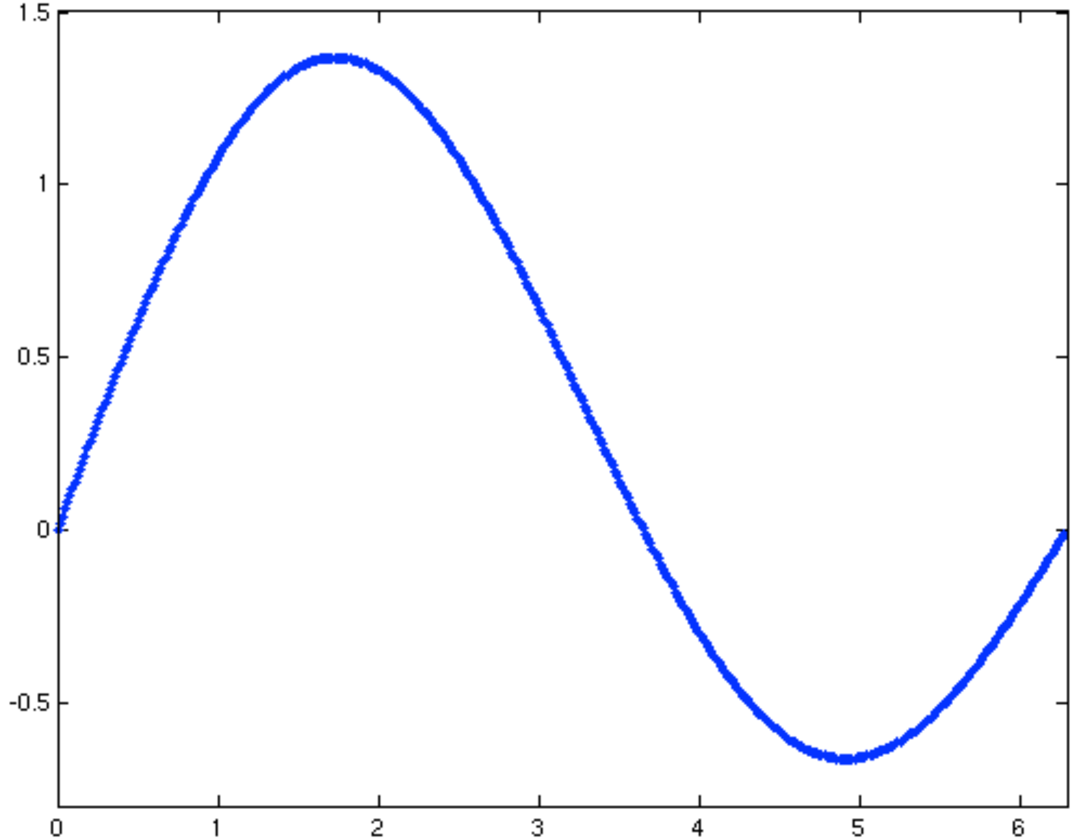
Stated differently: the maxima of the waves move faster than the minima and tend to reach them.

NOTE: this is a property of the equations and not of the initial data.

Even smooth initial data will lead to the appearance of shocks (in t>0) in the case of inviscid fluids.



$$\partial_t u + u\,\partial_x u = 0 \qquad u(x,0) = \sin x + \frac{1}{2}\sin\left(\frac{x}{2}\right)$$

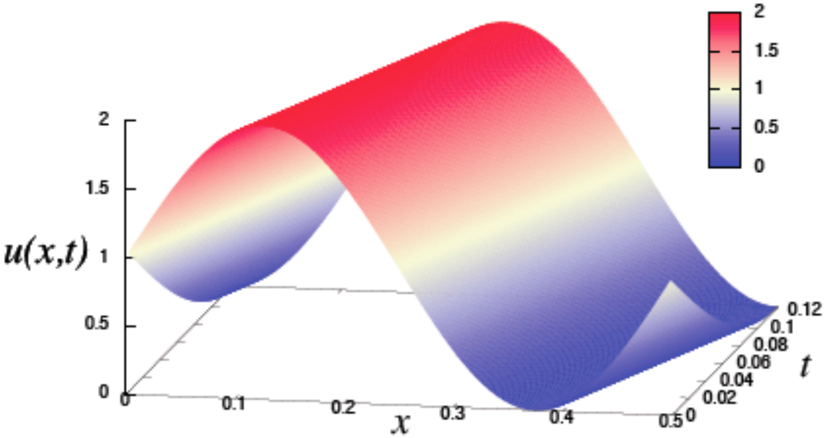


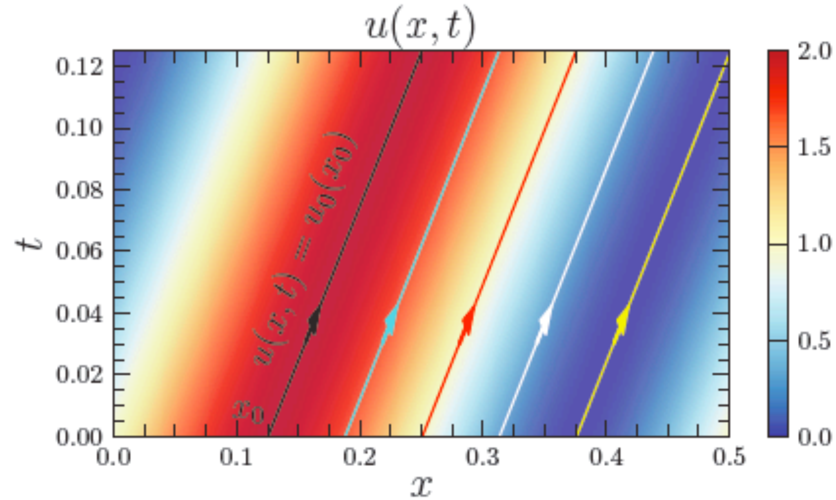
Credit: Balbás & Tadmor.

`CentPack` (high-resolution central schemes)

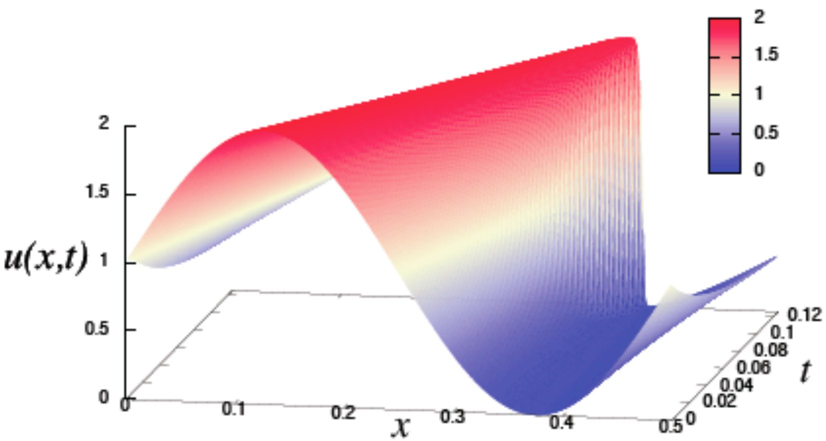
http://www.cscamm.umd.edu/centpack

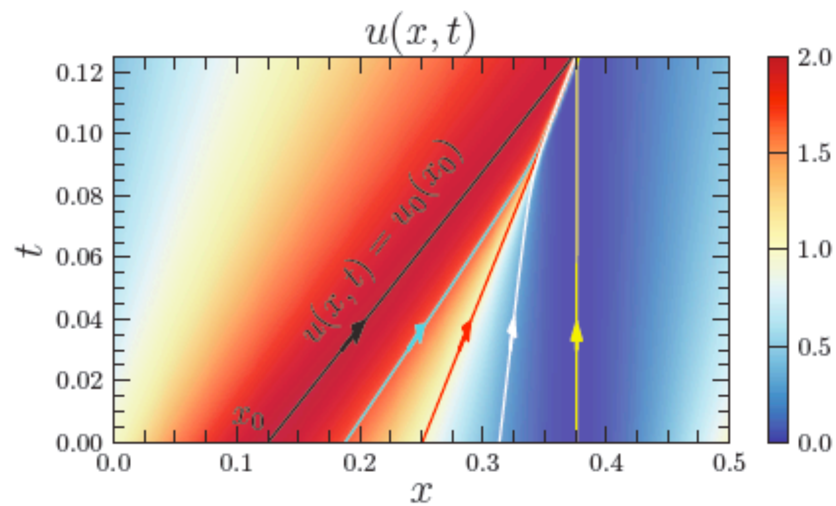
Advección vs Burgers (credit L. Rezzolla)





In the advection equation characteristic are parallel straight lines





In Burgers' equation characteristic are straight lines but not parallel. The "shock-steepening" leads to a shock and a caustic in the characteristic.

High-resolution methods: modified high-order finite-difference methods with appropriate amount of numerical dissipation in the vicinity of a discontinuity.

Quantity $\mathbf{u}_j^n$ is an approximation to $\mathbf{u}(x_j, t^n)$ but in the case of a conservation law it is preferable to view it as an approximation to the average within the numerical cell $[x_{j-1/2}, x_{j+1/2}]$

$$\mathbf{u}_j^n \sim \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{u}(x, t^n)\, dx$$

consistent with the integral form of the conservation law

For hyperbolic systems of conservation laws, schemes written in conservation form guarantee that the convergence (if it exists) is to one of the so-called weak solutions of the original system of equations (Lax-Wendroff theorem 1960).

A scheme written in conservation form reads:

$$\mathbf{u}_j^{n+1} = \mathbf{u}_j^n - \frac{\Delta t}{\Delta x}\left(\hat{\mathbf{f}}_{j+\frac{1}{2}}^n - \hat{\mathbf{f}}_{j-\frac{1}{2}}^n\right)$$

where $\hat{\mathbf{f}}$ is the numerical flux function

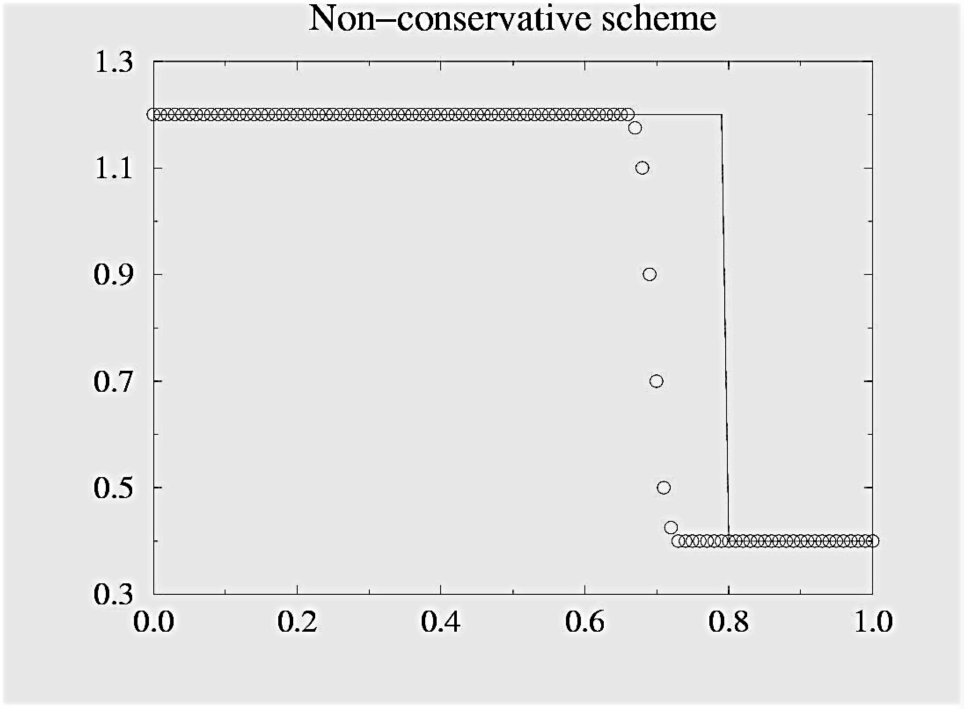
Example: Burgers equation with discontinuous initial data can be discretized using, e.g.

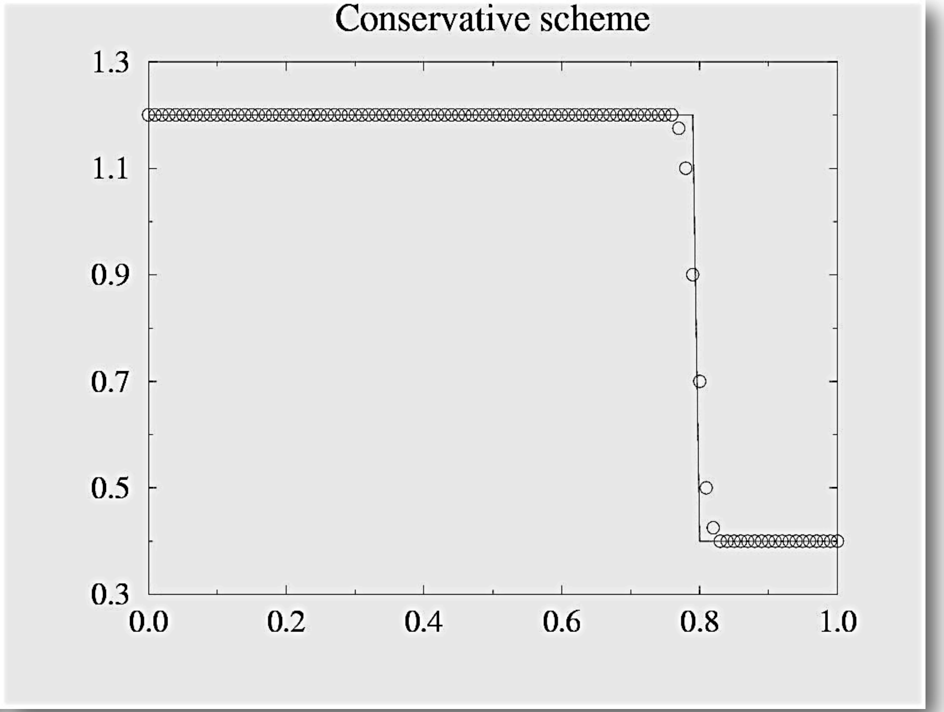
a **conservative upwind scheme:**

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(\frac{1}{2}(u_j^n)^2 - \frac{1}{2}(u_{j-1}^n)^2\right)$$

or using a **non-conservative upwind scheme:**

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x} u_j^n (u_j^n - u_{j-1}^n)$$





The conservation form of the scheme is ensured by starting with the integral version of the PDE in conservation form. By integrating the PDE within a spacetime computational cell

$$[x_{j-1/2}, x_{j+1/2}] \times [t^n, t^{n+1}]$$

the numerical flux function is an approximation to the time-averaged flux across the interface:

$$\hat{\mathbf{f}}_{j+1/2} \sim \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathbf{f}(\mathbf{u}(x_{j+1/2}, t))\, dt$$

The flux integral depends on the (unknown) solution at the numerical interfaces during the time step, $\mathbf{u}(x_{j+1/2}, t)$

Key idea (Godunov 1959): a possible procedure is to calculate this solution by solving Riemann problems at every cell interface.

$$\mathbf{u}(x_{j+1/2}, t) = \mathbf{u}(0, \mathbf{u}_j^n, \mathbf{u}_{j+1}^n)$$

Riemann solution for the left and right states along the ray x/t=0.

The Riemann problem

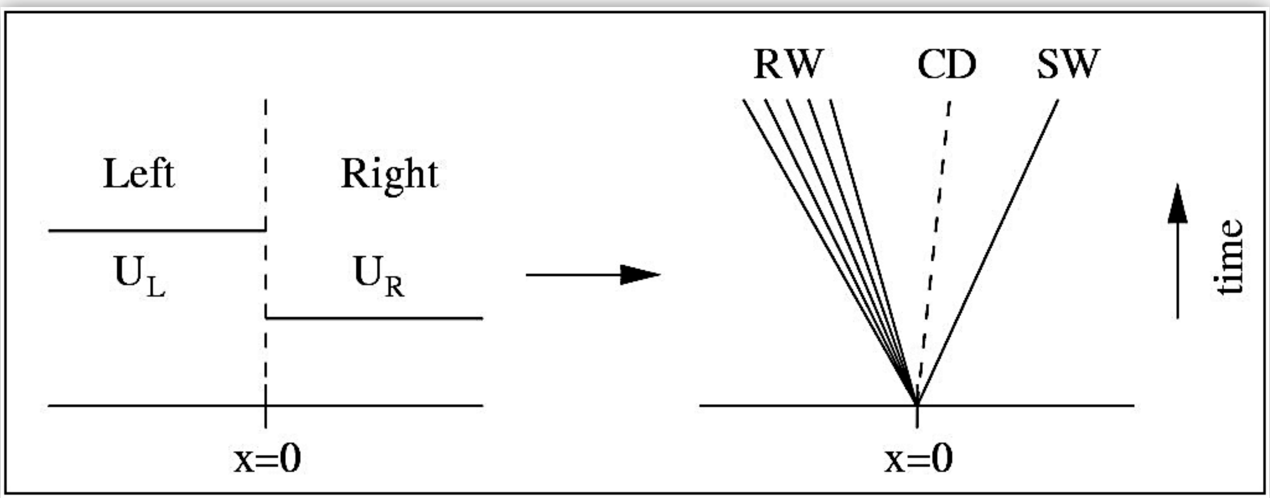
A Riemann problem is an IVP with discontinuous initial data:

$$\mathbf{u}_0 = \begin{cases} \mathbf{u}_L & \text{if x} < 0 \\ \mathbf{u}_R & \text{if x} > 0 \end{cases}$$

The Riemann problem is invariant under similarity transformations:

$$(x,t) \;\rightarrow\; (ax, at) \qquad\qquad a > 0$$

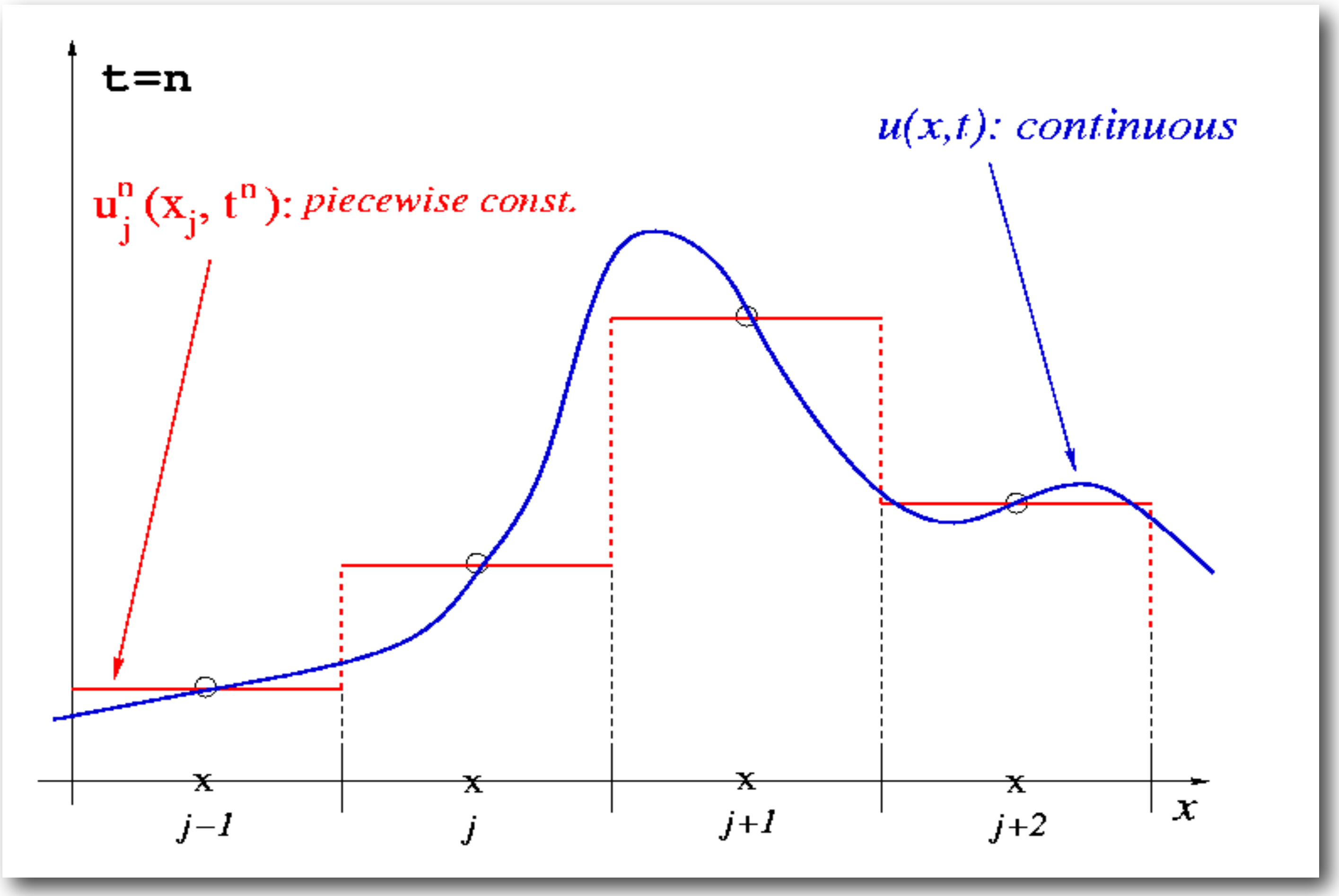
The solution is constant along the straight lines x/t=constant, and, hence, self-similar. It consists of **constant states separated by rarefaction waves** (continuous self-similar solutions of the differential equations), **shock waves, and contact discontinuities** (Lax 1972).



The incorporation of the **exact solution** of Riemann problems to compute the **numerical fluxes** of Euler's equations is due to **Godunov** (1959)

Why Riemann problems?

When a Cauchy problem described by a set of continuous PDEs is solved in a **discretized form** the numerical solution is **piecewise constant** (collection of local Riemann problems).

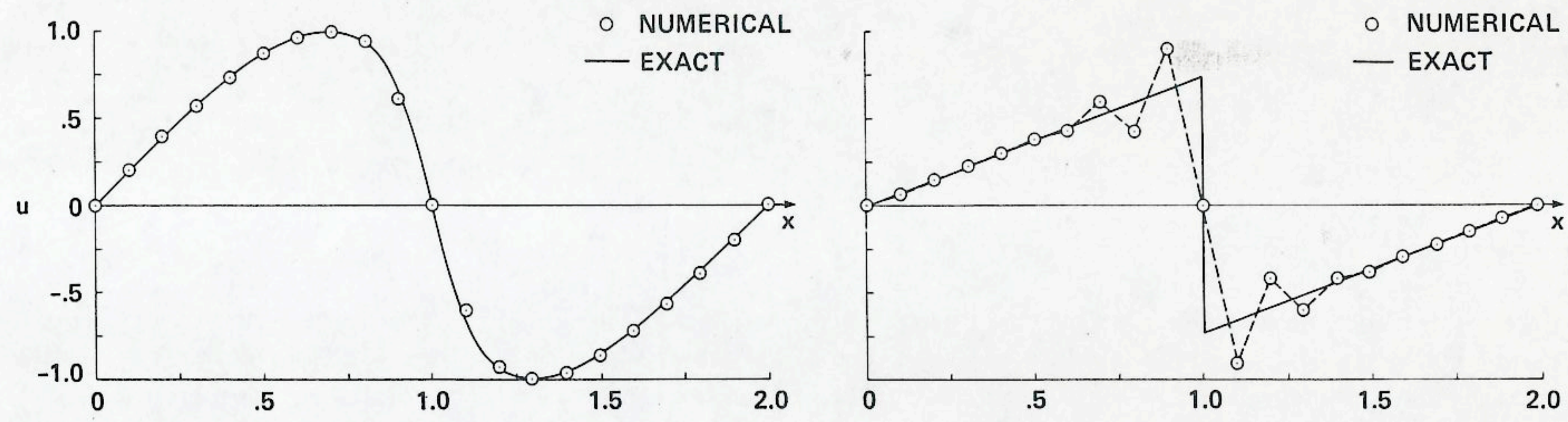


This is particularly problematic when solving the hydrodynamic equations (either Newtonian or relativistic) for compressible fluids.

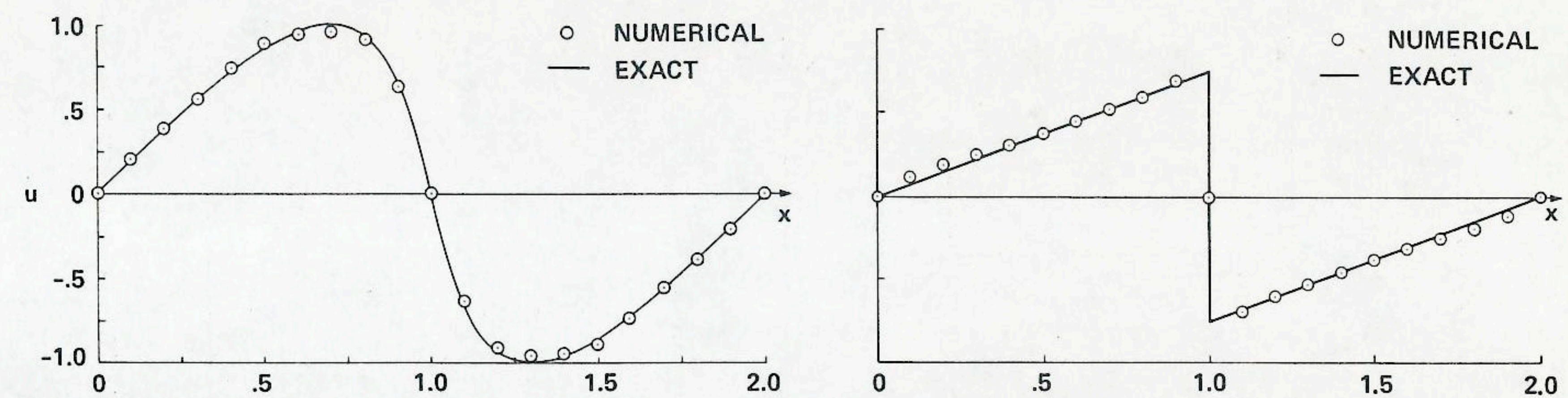
Their hyperbolic, nonlinear character produces discontinuous solutions in a finite time (shock waves, contact discontinuities) even from smooth initial data!

Any numerical scheme must be able to handle discontinuities in a satisfactory way.

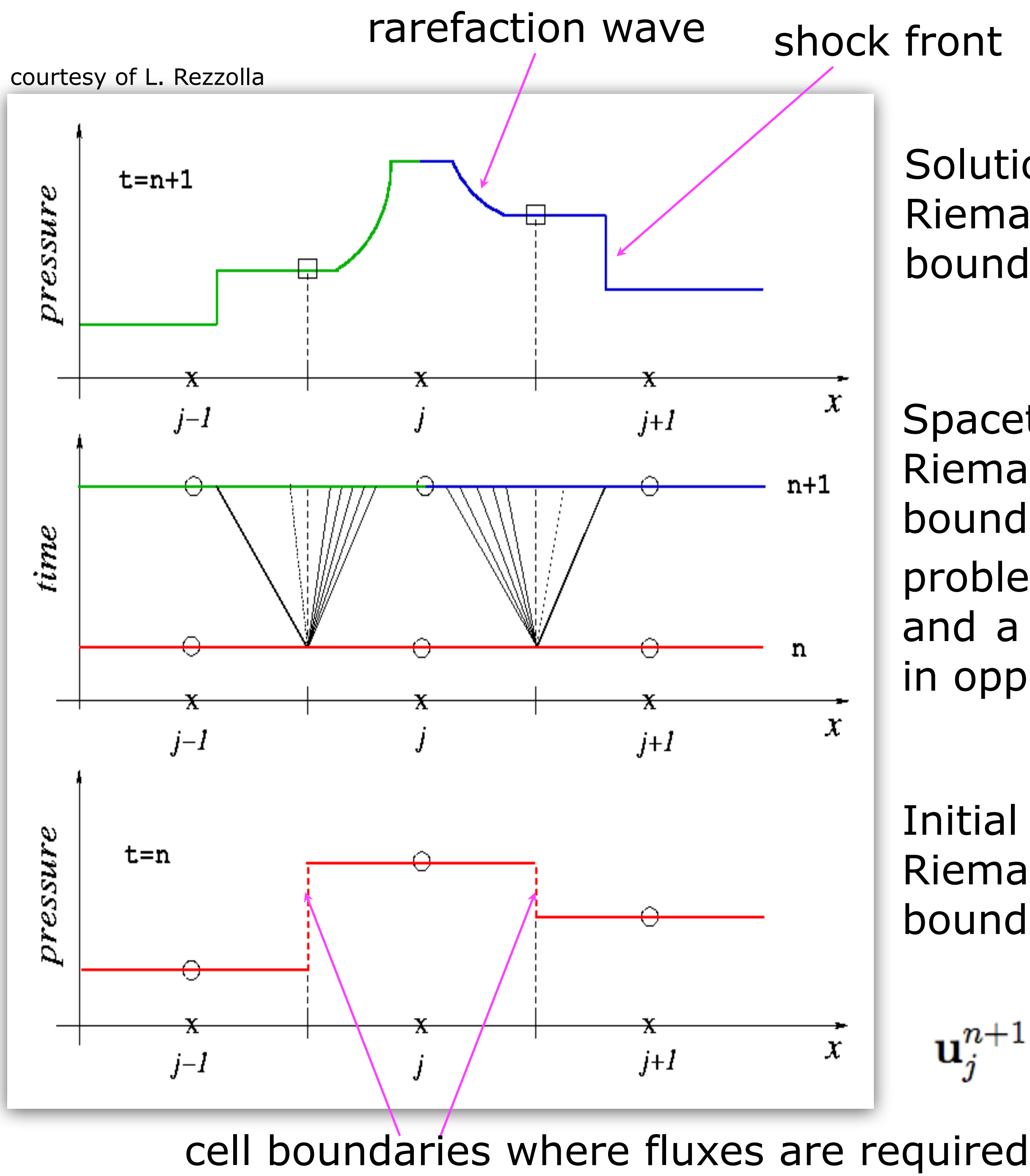
1. 1st order accurate schemes (Lax-Friedrich): Non-oscillatory but inaccurate across discontinuities (excessive diffusion)
2. (standard) 2nd order accurate schemes (Lax-Wendroff): Oscillatory across discontinuities
3. 2nd order accurate schemes with artificial viscosity
4. Godunov-type schemes (upwind High Resolution Shock Capturing schemes)



Lax-Wendroff numerical solution of Burger's equation at t=0.2 (left) and t=1.0 (right)



2nd order TVD numerical solution of Burger's equation at t=0.2 (left) and t=1.0 (right)



Solution at time n+1 of the two Riemann problems at the cell boundaries $x_{j+1/2}$ and $x_{j-1/2}$

Spacetime evolution of the two Riemann problems at the cell boundaries $x_{j+1/2}$ and $x_{j-1/2}$. Each problem leads to a shock wave and a rarefaction wave moving in opposite directions

Initial data at time n for the two Riemann problems at the cell boundaries $x_{j+1/2}$ and $x_{j-1/2}$

$$\mathbf{u}_j^{n+1} = \mathbf{u}_j^n - \frac{\Delta t}{\Delta x}\left(\hat{\mathbf{f}}^n_{j+\frac{1}{2}} - \hat{\mathbf{f}}^n_{j-\frac{1}{2}}\right)$$

Approximate Riemann solvers

In Godunov's method the structure of the Riemann solution is "lost" in the cell averaging process (1st order in space).

The exact solution of a RP is computationally expensive, particularly in multidimensions and for complicated EoS.

This motivated development of approximate (linearized) Riemann solvers.

Based on the exact solution of RP corresponding to a new system of equations obtained by a linearization of the original one (quasi-linear form). The spectral decomposition of the Jacobian matrices is on the basis of all solvers ("extending" ideas for linear systems).

$$\frac{\partial \vec{u}}{\partial t} + \frac{\partial \vec{f}}{\partial x} = 0 \Rightarrow \frac{\partial \vec{u}}{\partial t} + A\frac{\partial \vec{u}}{\partial x} = 0\,, \quad A = \frac{\partial \vec{f}}{\partial \vec{u}}$$

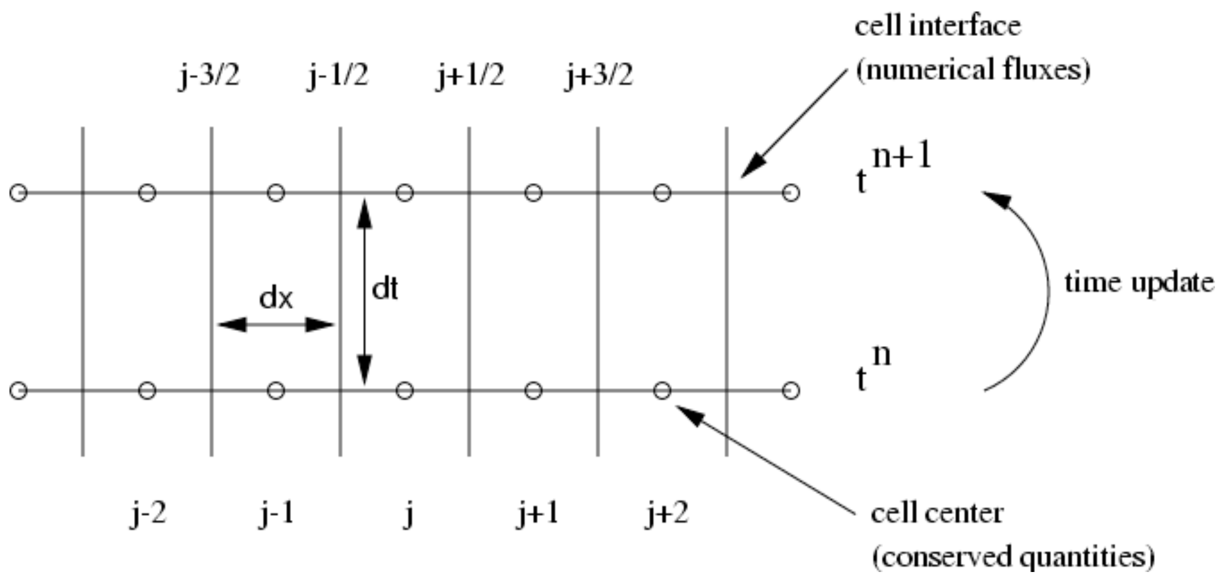
Approach followed by a subset of shock-capturing schemes, the so-called **Godunov-type methods** (Harten & Lax 1983; Einfeldt 1988).

See Martí & Müller (2003) for comprehensive discussion of approximate Riemann solvers for relativistic hydrodynamics.

Standard implementation of a HRSC scheme

1. Time update (MoL):

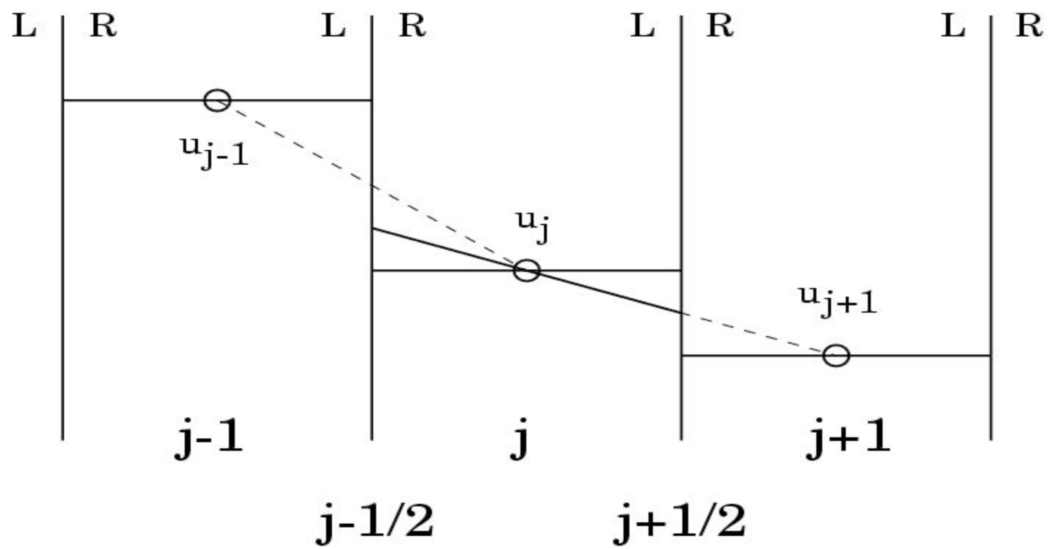
Algorithm in conservation form



$$\mathbf{u}_j^{n+1} = \mathbf{u}_j^n - \frac{\Delta t}{\Delta x}\left(\hat{\mathbf{f}}^n_{j+\frac{1}{2}} - \hat{\mathbf{f}}^n_{j-\frac{1}{2}}\right) + \Delta t\ \mathbf{S}_j^n$$

In practice: 2nd or 3rd order time accurate, conservative Runge-Kutta schemes (Shu & Osher 1989; MoL)

2. Cell reconstruction: Piecewise constant (Godunov), linear (MUSCL, MC, van Leer), parabolic (PPM, Colella & Woodward) interpolation procedures of state-vector variables from cell centers to cell interfaces.



3. Numerical fluxes: Approximate Riemann solvers (Roe, HLLE, Marquina). Explicit use of the spectral information of the system.

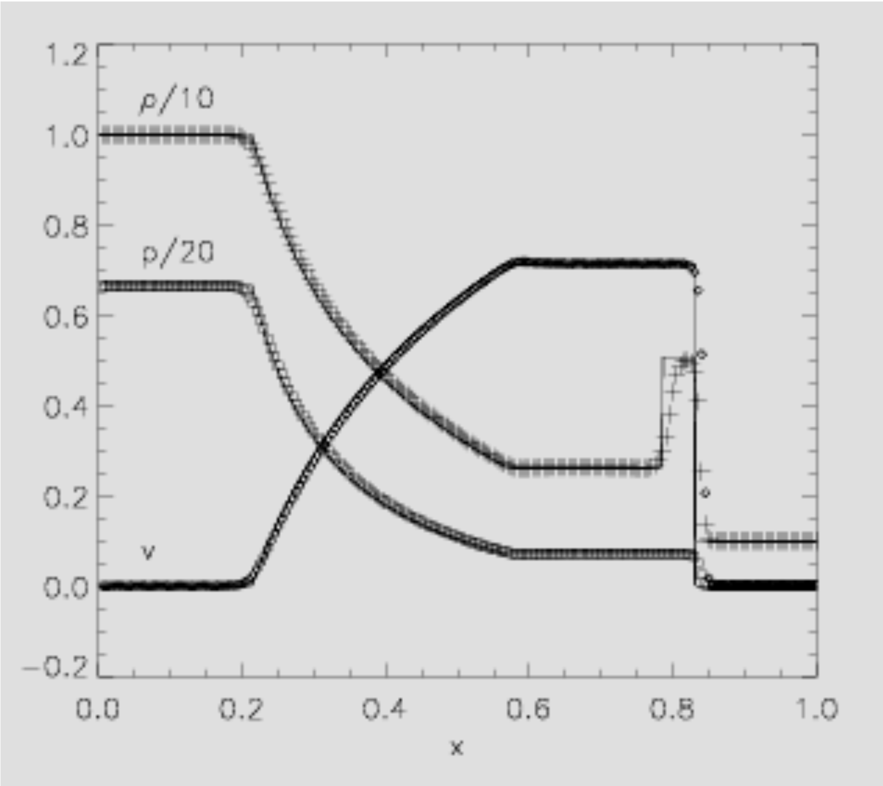
$$\hat{\mathbf{f}}_j = \frac{1}{2}\left[\mathbf{f}_j(\mathbf{w}_{\mathrm{R}}) + \mathbf{f}_j(\mathbf{w}_{\mathrm{L}}) - \sum_{i=1}^{5} |\tilde{\lambda}_i|\, \Delta\tilde{\mathbf{w}}_i\, \tilde{R}_i\right]$$

$$\Delta\mathbf{u} \equiv \mathbf{u}(\mathbf{w}_{\mathrm{R}}) - \mathbf{u}(\mathbf{w}_{\mathrm{L}}) = \sum_{i=1}^{5} \Delta\tilde{\mathbf{w}}_i\, \tilde{R}_i$$

High-resolution shock-capturing schemes

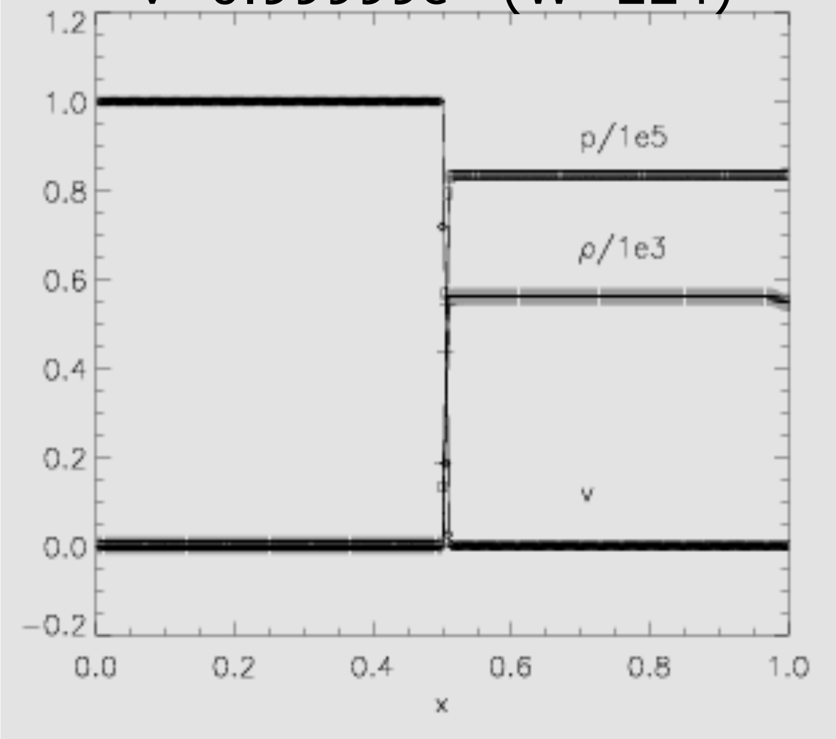
- **Stable and accurate shock profiles**
- **Accurate propagation speed of discontinuities**
- **Accurate numerical resolution of nonlinear features:** discontinuites, rarefaction waves, vortices, turbulence, etc

Shock tube test



Relativistic shock reflection

v=0.99999c (W=224)



Solution procedure of RMHD eqs

1. $$\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{\gamma}\mathbf{U}}{\partial t}+\frac{\partial\sqrt{-g}\mathbf{F}^i}{\partial x^i}\right)=\mathbf{S}$$

- Same HRSC as for the RHD equations
- Wave structure information obtained
- Primitive variable recovery more involved

2. $$\frac{\partial}{\partial x^i}(\sqrt{\gamma}B^i)=0$$

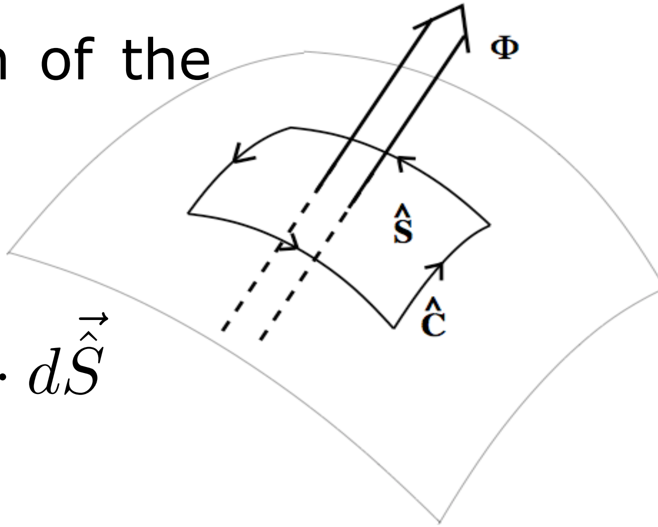
Divergence-free constraint not guaranteed to be satisfied numerically when updating the B-field with a HRSC scheme in conservation form.

Ad-hoc scheme must be used to update the B-field.

Main physical implication of divergence constraint is that magnetic flux through a closed surface is zero: essential to the constrained transport (CT) scheme (Evans & Hawley 1988, Tóth 2000).

$$\Phi_{\rm T} = \oint_{\hat{S}=\partial\hat{V}} \vec{B}^* \cdot d\vec{\hat{S}} = \int_{\hat{V}} \vec{\hat{\nabla}} \cdot \vec{B}^* d\hat{V} = 0$$

For any given surface, the time variation of the magnetic flux across the surface is:



(induction eq)

$$\frac{\partial\Phi}{\partial t} = \frac{\partial}{\partial t}\int_{\hat{S}} \vec{B}^* \cdot d\vec{\hat{S}} = \int_{\hat{S}} (\vec{\hat{\nabla}} \times \vec{v}^* \times \vec{B}^*) \cdot d\vec{\hat{S}}$$

$$= \int_{\hat{S}} (\vec{\hat{\nabla}} \times \vec{E}^*) \cdot d\vec{\hat{S}} = \int_{\hat{C}} \vec{E}^* \cdot d\vec{\hat{l}}$$

(Stokes theorem)

The magnetic flux through a surface can be computed as the line integral of the electric field along its boundary.

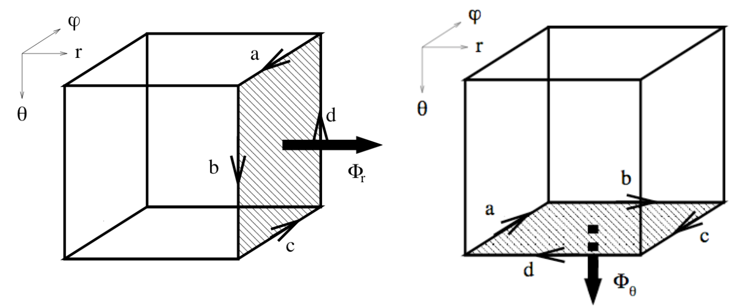
Flux-CT (implementation in axisymmetry)

Assumption: B-field components constant at cell surface
E-field components constant along cell edges

$$\frac{\partial \Phi_r}{\partial t} = \Delta \hat{S}_r \frac{\partial}{\partial t} B^{*r} = [E^*_\varphi \Delta \hat{l}_\varphi]_a - [E^*_\varphi \Delta \hat{l}_\varphi]_c$$

$$\frac{\partial \Phi_\theta}{\partial t} = \Delta \hat{S}_\theta \frac{\partial}{\partial t} B^{*\theta} = [E^*_\varphi \Delta \hat{l}_\varphi]_c - [E^*_\varphi \Delta \hat{l}_\varphi]_a$$

$$\frac{\partial \Phi_\varphi}{\partial t} = \Delta \hat{S}_\varphi \frac{\partial}{\partial t} B^{*\varphi} = [E^*_r \Delta \hat{l}_r]_c - [E^*_r \Delta \hat{l}_r]_a + [E^*_\theta \Delta \hat{l}_\theta]_d - [E^*_\theta \Delta \hat{l}_\theta]_b$$

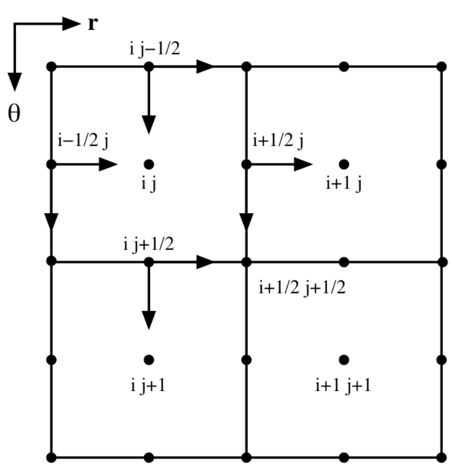


$$[E^*_\theta \Delta \hat{l}_\theta]_b = [E^*_\theta \Delta \hat{l}_\theta]_d$$

$$[E^*_r \Delta \hat{l}_r]_b = [E^*_r \Delta \hat{l}_r]_d$$

(axisymmetry condition)

Evolution eqs for B-field (CT scheme)



Polodial (r and θ) B-field components defined at cell interfaces (staggered grid)

Total magnetic flux through cell interfaces given by:

$$\Phi_{\mathrm{T}\ i\ j} = \Phi_{r\ i+\frac{1}{2}\ j} - \Phi_{r\ i-\frac{1}{2}\ j} + \Phi_{\theta\ i\ j+\frac{1}{2}} - \Phi_{\theta\ i\ j-\frac{1}{2}}$$

$$\left.\frac{\partial \Phi_{\mathrm{T}}}{\partial t}\right|_{i\ j} = 0$$

If initial data satisfy divergence constraint, it will be preserved during evolution

Discretisation:

$$\left.\frac{\partial B^{*r}}{\partial t}\right|_{i+\frac{1}{2}\ j} = \frac{\sin\theta_{j+\frac{1}{2}}\ E^*_{\varphi\ i+\frac{1}{2}\ j+\frac{1}{2}} - \sin\theta_{j-\frac{1}{2}}\ E^*_{\varphi\ i+\frac{1}{2}\ j-\frac{1}{2}}}{r_{i+\frac{1}{2}\ j}\ \Delta(\cos\theta)_j}$$

$$\left.\frac{\partial B^{*\theta}}{\partial t}\right|_{i\ j+\frac{1}{2}} = 2\ \frac{r_{i+\frac{1}{2}}\ E^*_{\varphi\ i+\frac{1}{2}\ j+\frac{1}{2}} - r_{i-\frac{1}{2}}\ E^*_{\varphi\ i-\frac{1}{2}\ j+\frac{1}{2}}}{\Delta r_i^2}$$

$$\left.\frac{\partial B^{*\varphi}}{\partial t}\right|_{i\ j} = \frac{2\Delta r_i}{\Delta\theta_j \Delta r_i^2}\left[E^*_{r\ i\ j+\frac{1}{2}} - E^*_{r\ i\ j-\frac{1}{2}}\right] - \frac{2}{\Delta r_i^2}\left[r_{i+\frac{1}{2}}\ E^*_{\theta\ i+\frac{1}{2}\ j} - r_{i-\frac{1}{2}}\ E^*_{\theta\ i-\frac{1}{2}\ j}\right]$$

E-field components computed from numerical fluxes of B-field conservation eqs. Done solving Riemann problems at cell interfaces. Procedure only valid for r and θ E-field components.

$$E^*_{r\ i\ j+\frac{1}{2}} = -\left[v^{*\theta}B^{*\varphi} - v^{*\varphi}B^{*\theta}\right]_{i\ j+\frac{1}{2}} = (\boldsymbol{F}^\theta)^\varphi_{i\ j+\frac{1}{2}}$$

$$E^*_{\theta\ i+\frac{1}{2}\ j} = -\left[v^{*\varphi}B^{*r} - v^{*r}B^{*\varphi}\right]_{i+\frac{1}{2}\ j} = -(\boldsymbol{F}^r)^\varphi_{i+\frac{1}{2}\ j}$$

$$E^*_{\varphi\ i+\frac{1}{2}\ j+\frac{1}{2}} = -\left[v^{*r}B^{*\theta} - v^{*\theta}B^{*r}\right]_{i+\frac{1}{2}\ j+\frac{1}{2}}$$

Balsara & Spicer (1999): ϕ component of the E-field computed from numerical fluxes in adjacent interfaces.

$$E^*_{\varphi\ i+\frac{1}{2}\ j+\frac{1}{2}} = -\frac{1}{4}\ \left[(\boldsymbol{F}^r)^\theta_{i\ j+\frac{1}{2}} + (\boldsymbol{F}^r)^\theta_{i+1\ j+\frac{1}{2}} - (\boldsymbol{F}^\theta)^r_{i+\frac{1}{2}\ j} - (\boldsymbol{F}^\theta)^r_{i+\frac{1}{2}\ j+1}\right]$$

Alternatives to CT method

For AMR codes based on unstructured grids and multiple coordinate systems,other schemes appear more suitable than the CT method to enforce B-field divergence-free constraint.

Projection method: involves solving an elliptic eq for a corrected B-field projected onto the subspace of zero divergence solutions by a linear operator.

B-field is decomposed into a curl and a gradient as

$$\vec{\mathcal{B}}^* = \nabla \times \vec{A} + \nabla \phi$$

whose divergence leads to an elliptic (Poisson) eq which can be solved for Φ.

$$\Delta \phi = \nabla \vec{\mathcal{B}}^*$$

B-field is next corrected according to $\vec{\mathcal{B}} = \vec{\mathcal{B}}^* - \nabla \phi$

[Alternative projection scheme can be implemented by taking the curl of the above equation and solving for the vector potential.]

Hyperbolic divergence cleaning: based on the introduction of an additional scalar field which is coupled to the magnetic field by a gradient term in the induction equation.

The scalar field is computed by adding an additional constraint hyperbolic equation given by

$$\frac{d\psi}{dt} = -c_h^2(\nabla\vec{B})$$

Divergence errors propagated off the grid in a wave-like manner with characteristic speed c_h.

CT and divergence cleaning implemented on the ET.

Homework #1: download the open source codes from the Living Review article by Martí & Müller (2003) and use it to solve various shock tube problems with the special relativistic hydrodynamics equations.

"Numerical Hydrodynamics in Special Relativity"
José Maria Martí and Ewald Müller
http://www.livingreviews.org/lrr-2003-7

Program RIEMANN

This program computes the solution of a 1D relativistic Riemann problem.

Program rPPM

This program simulates 1D relativistic flows in Cartesian coordinates using the exact Riemann solver and PPM reconstruction.

Homework #2: download the open source codes from the Living Review article by Martí & Müller (2015) and use it to solve 1D Riemann problems and 1D flows with the relativistic MHD equations.

"Grid-based methods in relativistic hydrodynamics and MHD”
José Maria Martí and Ewald Müller
http://www.livingreviews.org/lrca-2015-3

Program riemann_rmhd

This program computes the solution of a 1D relativistic Riemann problem in RMHD. Developed by Giacomazzo & Rezzolla (2006)

Program rmhd_1d

This program simulates 1D relativistic flows in Cartesian coordinates using different techniques. Standard RMHD test suite included.